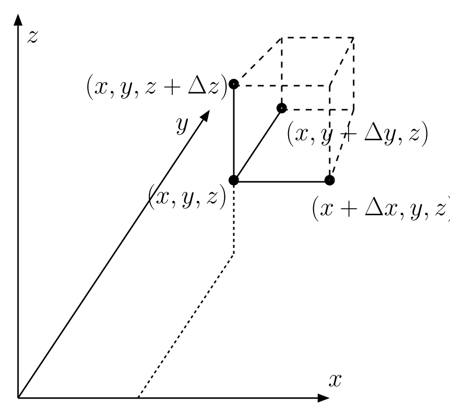
4 Divergence and curl

Expressing the total charge Q_V contained in a volume V as a 3D volume integral of charge density $\rho(\mathbf{r})$, we can write *Gauss's law* examined during the last few lectures in the general form

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV.$$

This equation asserts that the flux of displacement $\mathbf{D} = \epsilon_o \mathbf{E}$ over any closed surface S equals the net electrical charge contained in the enclosed volume V — only the charges included within V affect the flux of $\mathbf{D}$ over surface S, with charges outside surface S making no net contribution to the surface integral $\oint_S \mathbf{D} \cdot d\mathbf{S}$.



- Gauss's law stated above holds true everywhere in space over all surfaces S and their enclosed volumes V, large and small.

- Application of Gauss's law to a small volume $\Delta V = \Delta x \Delta y \Delta z$ surrounded by a cubic surface ΔS of six faces, leads, in the limit of vanishing Δx, Δy, and Δz, to the differential form of Gauss's law expressed in terms of a **divergence operation** to be reviewed next:

  – Given a sufficiently small volume $\Delta V = \Delta x \Delta y \Delta z$, we can assume that

$$\int_{\Delta V} \rho dV \approx \rho \Delta x \Delta y \Delta z.$$

– Again under the same assumption

$$\oint_S \mathbf{D}\cdot d\mathbf{S} \approx (D_{x|2}-D_{x|1})\Delta y\Delta z+(D_{y|4}-D_{y|3})\Delta x\Delta z+(D_{z|6}-D_{z|5})\Delta x\Delta y$$

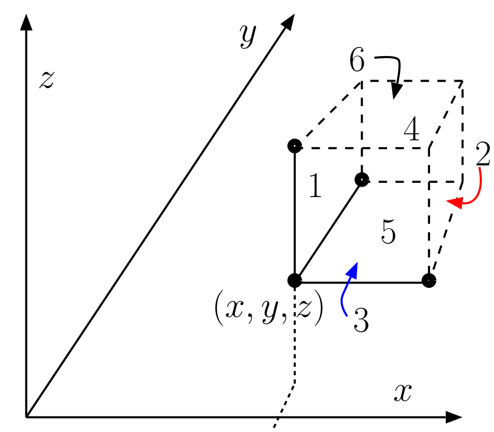
with reference to displacement vector components like $D_{x|2}$ shown on cubic surfaces depicted in the margin. Gauss's law demands the equality of the two expressions above, namely (after dividing both sides by $\Delta x\Delta y\Delta z$)

$$\frac{D_{x|2}-D_{x|1}}{\Delta x}+\frac{D_{y|4}-D_{y|3}}{\Delta y}+\frac{D_{z|6}-D_{z|5}}{\Delta z}\approx \rho,$$

in the limit of vanishing Δx, Δy, and Δz. In that limit, we obtain

$$\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}=\rho,$$

which is known as **differential form of Gauss's law**.



A more compact way of writing this result is

$$\nabla\cdot\mathbf{D}=\rho,$$

where the operator

$$\nabla\equiv(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}),$$

known as *del*, is applied on the displacement vector

$$\mathbf{D}=(D_x,D_y,D_z)$$

following the usual dot product rules, except that the product of $\frac{\partial}{\partial x}$ and D_x, for instance, is treated as a partial derivative $\frac{\partial D_x}{\partial x}$. In the left side above

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \quad \text{(divergence of } \mathbf{D}\text{)}$$

is known as **divergence** of **D**.

Example 1: Find the divergence of $\mathbf{D} = \hat{x}5x + \hat{y}12$ C/m^2

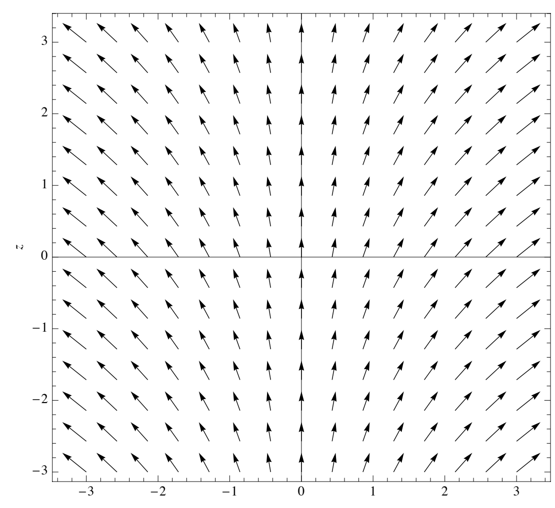
Solution: In this case

$$D_x = 5x, \quad D_y = 12, \quad \text{and} \quad D_z = 0.$$

Therefore, divergence of **D** is

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \\ &= \frac{\partial}{\partial x}(5x) + \frac{\partial}{\partial y}(12) + \frac{\partial}{\partial z}(0) \\ &= 5 + 0 + 0 = 5\,\frac{\mathrm{C}}{\mathrm{m}^3}. \end{aligned}$$

Note that the divergence of vector **D** is a scalar quantity which is the volumetric charge density in space as a consequence of Gauss's law (in differential form).



Example 2: Find the divergence $\nabla \cdot \mathbf{E}$ of electric field vector

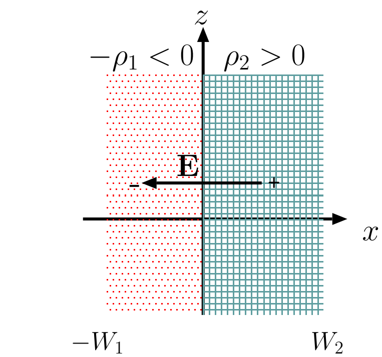
$$\mathbf{E} = \begin{cases} -\hat{x}\frac{\rho_1(x+W_1)}{\epsilon_o}, & \text{for } -W_1 < x < 0 \\ \hat{x}\frac{\rho_2(x-W_2)}{\epsilon_o}, & \text{for } 0 < x < W_2 \\ 0, & \text{otherwise,} \end{cases}$$

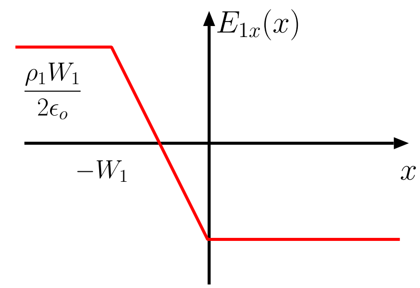
from Example 4, last lecture (see margin figures).

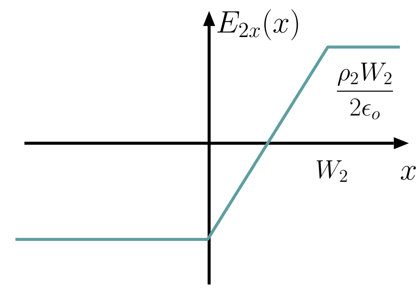
Solution: In this case $E_y = E_z = 0$, and therefore the divergence of $\mathbf{E}$ is

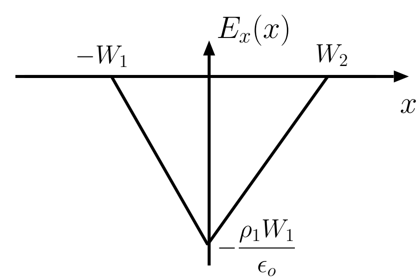
$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} = \frac{\partial}{\partial x}\begin{cases} -\frac{\rho_1(x+W_1)}{\epsilon_o}, \\ \frac{\rho_2(x-W_2)}{\epsilon_o}, \\ 0, \end{cases} = \begin{cases} -\frac{\rho_1}{\epsilon_o}, & \text{for } -W_1 < x < 0 \\ \frac{\rho_2}{\epsilon_o}, & \text{for } 0 < x < W_2 \\ 0, & \text{otherwise,} \end{cases},$$

which provides us with $\rho(\mathbf{r})/\epsilon_o$ of Example 4 from last lecture (in accordance with Gauss's law).









- Summarizing the results so far, Gauss's law can be expressed in *integral* as well as *differential* forms given by

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV \quad \Leftrightarrow \quad \nabla \cdot \mathbf{D} = \rho.$$

  – The equivalence of integral and differential forms implies that (after integrating the differential form of the equation on the right over volume V on both sides)

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{D}\, dV$$

which you may recall as the **divergence theorem** from MATH 241. **Divergence thm.**

– Note that according to divergence theorem, we can interpret **divergence as flux per unit volume**.

– We can also think of divergence as a special type of a derivative applied to vector functions which produces non-zero scalar results (at each point in space) when the vector function has components which change in the direction they point.

  ◦ A second type of vector derivative known as **curl** which we review next complements the divergence in the sense that these two types of vector derivatives collectively contain maximal information about vector fields that they operate on:

Given their curl and divergences, vector fields can be uniquely reconstructed in regions V of 3D space provided they are known at the bounding surface S of region V, however large (even infinite) S and V may be — this is known as **Helmholtz theorem** (proof outlined in Lecture 7).

- The **curl** of a vector field $\mathbf{E} = \mathbf{E}(x, y, z)$ is defined, in terms of the del operator ∇, like a cross product

$$\nabla \times \mathbf{E} \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (E_x, E_y, E_z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \qquad \text{(curl of } \mathbf{E}\text{)}$$

$$= \hat{x}(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) - \hat{y}(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}) + \hat{z}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}).$$

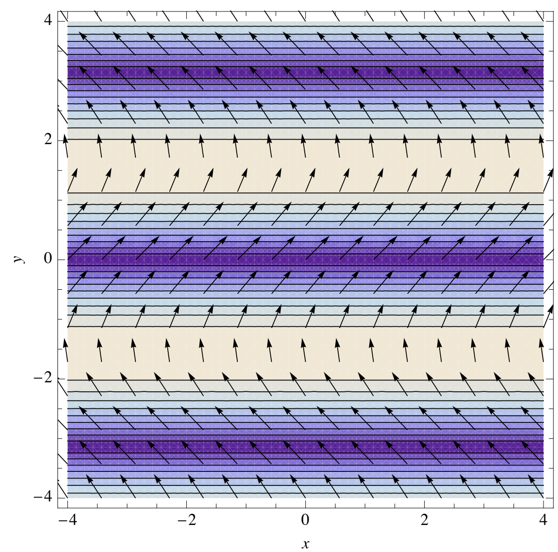
Example 3: Find the curl of the vector field

$$\mathbf{E} = \hat{x}\cos y + \hat{y}1$$

Solution: The curl is

$$\begin{aligned} \nabla \times \mathbf{E} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & 1 & 0 \end{vmatrix} \\ &= \hat{x}(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}1) - \hat{y}(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}\cos y) + \hat{z}(\frac{\partial}{\partial x}1 - \frac{\partial}{\partial y}\cos y) \\ &= \hat{x}0 - \hat{y}0 + \hat{z}(0 + \sin y) = \hat{z}\sin y \end{aligned}$$

which is another vector field.



The diagram in the margin depicts $\mathbf{E} = \hat{x}\cos y + \hat{y}1$ as a vector map superposed upon a density plot of $|\nabla \times \mathbf{E}| = |\hat{z}\sin y| = |\sin y|$ indicating the strength of the curl vector $\nabla \times \mathbf{E}$ (light color corresponds large magnitude).

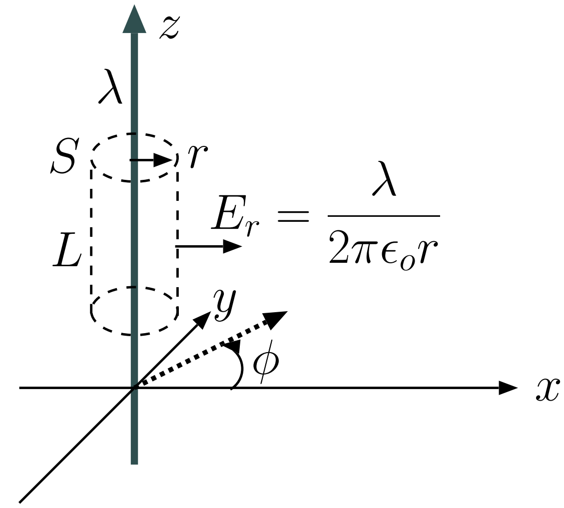
It is apparent that curl $\nabla \times \mathbf{E}$ is stronger in those regions where $\mathbf{E}$ is rapidly varying in directions orthogonal to the direction of $\mathbf{E}$ itself.

- As the above example demonstrates the curl of a vector field is in general another vector field.
  - The only exception is if the curl is identically 0 at all positions $\mathbf{r} = (x, y, z)$!
    - In that case, i.e., if $\nabla \times \mathbf{E} = 0$, vector field $\mathbf{E}$ is said to be **curl-free**.

IMPORTANT FACT: All static electric fields E, obtained from Coulomb's law, and satisfying Gauss's law $\nabla \cdot \mathbf{D} = \rho$ with static charge densities $\rho = \rho(\mathbf{r})$, are also found to be *curl-free* without exception.

- The proof of curl-free nature of static electric fields can be given by first showing that Coulomb field of a static charge is curl-free, and then making use of the superposition principle along with the fact that the curl of a sum must be the sum of curls — like differentiation, "taking curl" is a linear operation.
  - You should try to show that $\nabla \times \mathbf{E} = 0$ with the Coulomb field of a point charge Q located at the origin.

- The calculation is slightly more complicated than the following example (although similar in many ways) where we show that the static electric field of an infinite line charge is curl-free.



Example 4: Recall that the static field of a line charge λ distributed on the z-axis is

$$\mathbf{E}(x,y,z) = \hat{r}\frac{\lambda}{2\pi\epsilon_o r},$$

where

$$r^2 = x^2 + y^2 \ \text{ and } \ \hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi = (\frac{x}{r}, \frac{y}{r}, 0).$$

Show that field $\mathbf{E}$ satisfies the condition $\nabla \times \mathbf{E} = 0$.

Solution: Clearly, we can express vector $\mathbf{E}$ as

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_o}(\frac{x}{r^2}, \frac{y}{r^2}, 0).$$

Since the components $\frac{x}{r^2}$ and $\frac{y}{r^2}$ of the vector are independent of z, the corresponding curl can be expanded as

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \frac{\lambda}{2\pi\epsilon_o}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^2} & \frac{y}{r^2} & 0 \end{vmatrix} = \frac{\lambda}{2\pi\epsilon_o}\hat{z}(\frac{\partial}{\partial x}\frac{y}{r^2} - \frac{\partial}{\partial y}\frac{x}{r^2}).$$

But,

$$\frac{\partial}{\partial x}\frac{y}{r^2} - \frac{\partial}{\partial y}\frac{x}{r^2} = y\frac{\partial}{\partial x}\frac{1}{r^2} - x\frac{\partial}{\partial y}\frac{1}{r^2} = y\frac{-2x}{r^4} - x\frac{-2y}{r^4} = 0,$$

so $\nabla \times \mathbf{E} = 0$ as requested.