5 Curl-free fields and electrostatic potential

- Mathematically, we can generate a curl-free vector field $\mathbf{E}(x, y, z)$ as

$$\mathbf{E} = -(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}),$$

  by taking the **gradient** of any scalar function $V(\mathbf{r}) = V(x, y, z)$. The gradient of $V(x, y, z)$ is defined to be the vector

$$\nabla V \equiv (\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}),$$

  pointing in the direction of increasing V; in abbreviated notation, curl-free fields $\mathbf{E}$ can be indicated as

$$\mathbf{E} = -\nabla V.$$

  - **Verification:** Curl of vector ∇V is

$$\nabla \times (\nabla V) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix} = \hat{x}0 - \hat{y}0 - \hat{z}0 = 0.$$

  - If $\mathbf{E} = -\nabla V$ represents an **electrostatic field**, then V is called the **electrostatic potential**.
    - Simple dimensional analysis indicates that units of electrostatic potential must be volts (V).

– The prescription $\mathbf{E} = -\nabla V$, including the minus sign (optional, but taken by convention in electrostatics), ensures that electrostatic field $\mathbf{E}$ points from regions of "high potential" to "low potential" as illustrated in the next example.

Electrostatic fields E point from regions of "high V" to "low V"

Example 1: Given an electrostatic potential

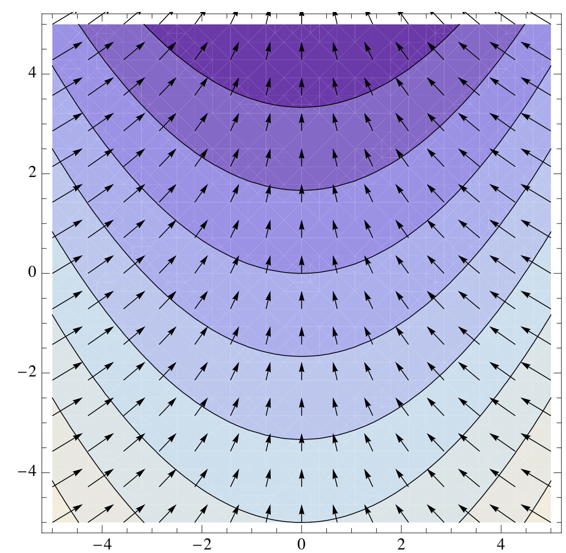
$$V(x, y, z) = x^2 - 6y\,\mathrm{V}$$

in a certain region of space, determine the corresponding electrostatic field $\mathbf{E} = -\nabla V$ in the same region.

Solution: The electrostatic field is

$$\mathbf{E} = -\nabla(x^2 - 6y) = -(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(x^2 - 6y) = (-2x, 6, 0) = -\hat{x}\,2x + \hat{y}6\,\mathrm{V/m}.$$

Note that this field is directed from regions of high potential to low potential. Also note that electric field vectors are perpendicular everywhere to "equipotential" contours.



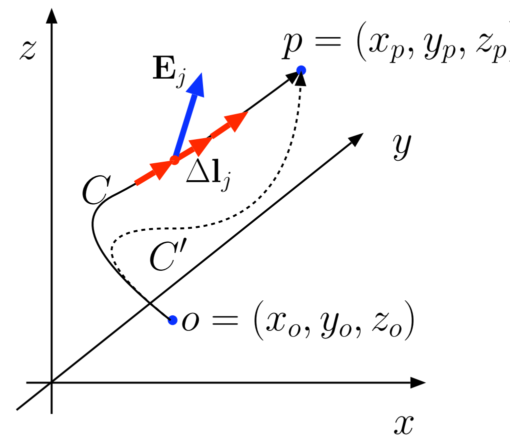
Light colors indicate "high V" dark colors "low V"

Given an electrostatic potential $V(x, y, z)$, finding the corresponding electrostatic field $\mathbf{E}(x, y, z)$ is a straightforward procedure (taking the negative gradient) as already illustrated in Example 1.

The reverse operation of finding $V(x, y, z)$ from a given $\mathbf{E}(x, y, z)$ can be accomplished by performing a **vector line integral**

$$\int_o^p \mathbf{E} \cdot d\mathbf{l}$$

in 3D space, since, as shown below, such integrals are "path independent" for curl-free fields $\mathbf{E} = -\nabla V$.



- The **vector line integral**

$$\int_o^p \mathbf{E} \cdot d\mathbf{l}$$

  over an integration path C extending from a point $o = (x_o, y_o, z_o)$ in 3D space to some other point $p = (x_p, y_p, z_p)$ is *defined to be*

  - the limiting value of the sum of dot products $\mathbf{E}_j \cdot \Delta\mathbf{l}_j$ computed over all sub-elements of path C having incremental lengths $|\Delta\mathbf{l}_j|$ and unit vectors $\Delta\mathbf{l}_j/|\Delta\mathbf{l}_j|$ directed from o towards p — the limiting value is obtained as all $|\Delta\mathbf{l}_j|$ approach zero (i.e., with increasingly finer subdivision of C into $|\Delta\mathbf{l}_j|$ elements).

- Computation of the integral (see example below) involves the use of infinitesimal displacement vectors

$$d\mathbf{l} = \hat{x}dx + \hat{y}dy + \hat{z}dz = (dx, dy, dz)$$

  and vector dot product

$$\mathbf{E} \cdot d\mathbf{l} = (E_x, E_y, E_z) \cdot (dx, dy, dz) = E_x dx + E_y dy + E_z dz.$$

  The integral

$$\int_o^p \mathbf{E} \cdot d\mathbf{l} = \int_o^p (E_x dx + E_y dy + E_z dz)$$

will in general be *path dependent* except for when $\mathbf{E}$ is curl-free.

Example 2: The field $\mathbf{E} = \hat{x}y \pm \hat{y}x$ is curl-free with the $+$ sign, but not with $-$ as verified below by computing $\nabla \times \mathbf{E}$. Calculate the line integral of $\mathbf{E}$ (for both signs, $\pm$) from point $o = (0, 0, 0)$ to point $p = (1, 1, 0)$ for two different paths C going through points $u = (0, 1, 0)$ and $l = (1, 0, 0)$, respectively (see margin).

Solution: First we note that

$$\nabla \times (\hat{x}y \pm \hat{y}x) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & \pm x & 0 \end{vmatrix} = \hat{z}(\pm 1 - 1)$$

which confirms that $\mathbf{E} = \hat{x}y \pm \hat{y}x$ is curl-free with with $+$ sign, but not with $-$. In either case, the integral to be performed is

$$\int_o^p \mathbf{E} \cdot d\mathbf{l} = \int_o^p (E_x dx + E_y dy + E_z dz) = \int_o^p (y\,dx \pm x\,dy).$$

For the first path C_u going through $u = (0, 1, 0)$, we have

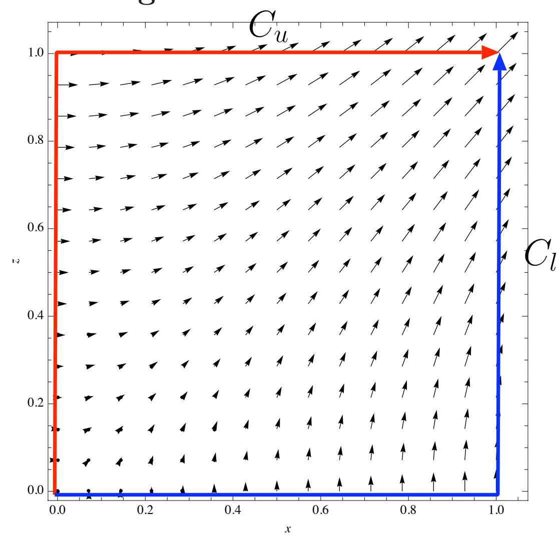
$$\int_o^p (y\,dx \pm x\,dy) = \int_{y=0}^1 (\pm x)\,dy_{|x=0} + \int_{x=0}^1 y\,dx_{|y=1} = 0 + 1 = 1.$$

For the second path C_l going through $l = (1, 0, 0)$, we have

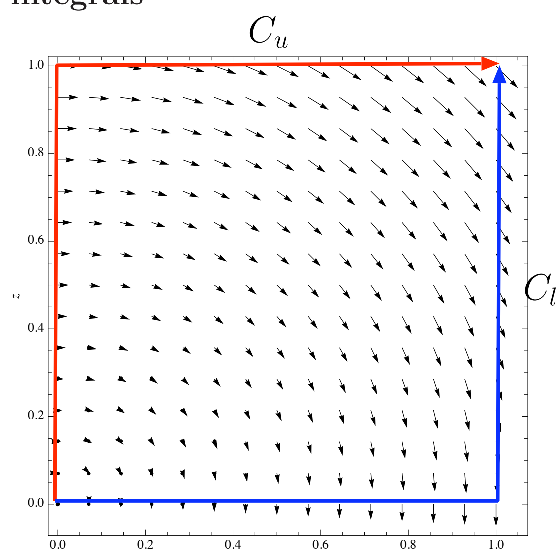
$$\int_o^p (y\,dx \pm x\,dy) = \int_{x=0}^1 y\,dx_{|y=0} \pm \int_{y=0}^1 x\,dy_{|x=1} = 0 \pm 1 = \pm 1.$$

Clearly, the result shows that the line integral $\int_o^p \mathbf{E} \cdot d\mathbf{l}$ is *path independent* for $\mathbf{E} = \hat{x}y + \hat{y}x$ which is curl-free, and path dependent for $\mathbf{E} = \hat{x}y - \hat{y}x$ in which case $\nabla \times \mathbf{E} \neq 0$.

Curl-free: path-independent line integrals



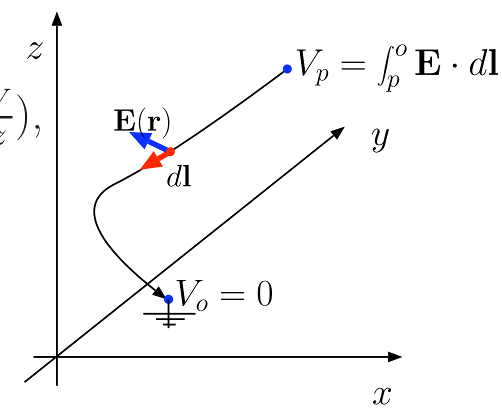
"Curly": path-dependent line integrals



- The **mathematical reason** why curl-free fields have path-independent line integrals is because in those occasions the integrals can be written in terms of **exact differentials**:

  – for curl-free $\mathbf{E} = \hat{x}y + \hat{y}x$ we have $\mathbf{E}\cdot d\mathbf{l}$ as an *exact differential* $ydx + xdy = d(xy)$ of the function xy, in which case $\int_o^p \mathbf{E}\cdot d\mathbf{l} = xy|_o^p = (1\cdot 1 - 0\cdot 0) = 1$ over all paths.

  – for $\mathbf{E} = \hat{x}y - \hat{y}x$ with $\nabla\times\mathbf{E} = -2\hat{z} \neq 0$, on the other hand, $\mathbf{E}\cdot d\mathbf{l} = ydx - xdy$ does not form an exact differential $-dV$, and thus there is no path-independent integral $-V|_o^p$, nor an underlying potential function V.



$\mathbf{E}\cdot d\mathbf{l}$ is *guaranteed* to be an exact differential if $\mathbf{E} = -\nabla V = (-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z})$, since in that case the differential of $V(x,y,z)$, namely

$$dV \equiv \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz, \text{ is precisely } -E_x dx - E_y dy - E_z dz = -\mathbf{E}\cdot d\mathbf{l}.$$

  – In that case

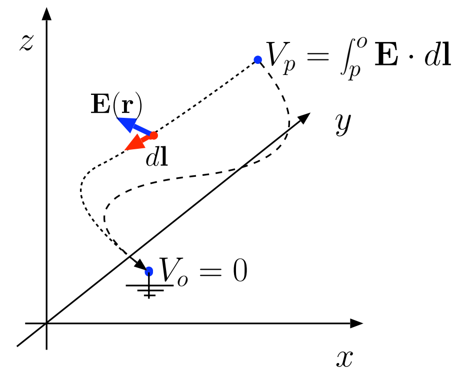
$$\int_o^p \mathbf{E}\cdot d\mathbf{l} = -\int_p^o \mathbf{E}\cdot d\mathbf{l} = \int_p^o dV = V|_p^o = V_o - V_p$$

  is independent of integration path; thus, if we we call o the "ground", and set $V_o = 0$, then

$$V_p = \int_p^o \mathbf{E}\cdot d\mathbf{l}$$

  denotes the potential drop from (any) point p to ground o.

- The **physical reason** why this integral formula for potential V_p works with *any* integration path is the principle of **energy conservation**:
  - integral $\int_p^o \mathbf{E}\cdot d\mathbf{l}$ represents the **work done** by field **E per unit charge** moved from p to o, so if the line integral were path-dependent there would be ways of creating net energy by making a charge q follow special paths within the electrostatic field **E**, in violation of the general principle of energy conservation (that permits energy conversion but not creation or destruction).



As long as E is curl-free, line integral is path-independent and produces the voltage drop from point p to "ground" o.

Example 3: Given that $V_o = V(0,0,0) = 0$ and

$$\mathbf{E} = 2x\hat{x} + 3z\hat{y} + 3(y+1)\hat{z}\,\frac{\mathrm{V}}{\mathrm{m}},$$

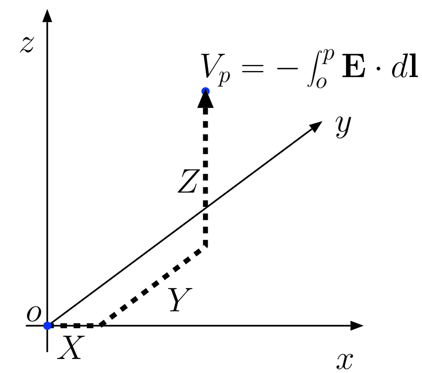
determine the electrostatic potential $V_p = V(X,Y,Z)$ at point $p = (X,Y,Z)$ in volts.

Solution: Assuming that the field is curl-free (it is), so that any integration path can be used, we find that

$$\begin{aligned} V_p &= \int_p^o \mathbf{E}\cdot d\mathbf{l} = -\int_o^p \mathbf{E}\cdot d\mathbf{l} = -\int_o^p (2x\,dx + 3z\,dy + 3(y+1)\,dz) \\ &= -\int_0^X 2x\,dx_{|y,z=0} - \int_0^Y 3z\,dy_{|x=X,z=0} - \int_0^Z 3(y+1)\,dz_{|x=X,y=Y} \\ &= -X^2 - 0 - 3(Y+1)Z. \end{aligned}$$

This implies

$$V(x,y,x) = -x^2 - 3(y+1)z\,\mathrm{V}.$$



Note that

$$\begin{aligned}-\nabla(-x^2-3(y+1)z) &= \nabla(x^2+3(y+1)z)\\ &= \hat{x}2x+\hat{y}3z+\hat{z}3(y+1)\end{aligned}$$

yields the original field $\mathbf{E}$, which is an indication that $\mathbf{E}$ is indeed curl-free.

Example 5: According to Coulomb's law electrostatic field of a proton with charge $Q=e$ (where $-e$ is electronic charge) located at the origin is given as

$$\mathbf{E}=\frac{e}{4\pi\epsilon_o r^2}\hat{r},$$

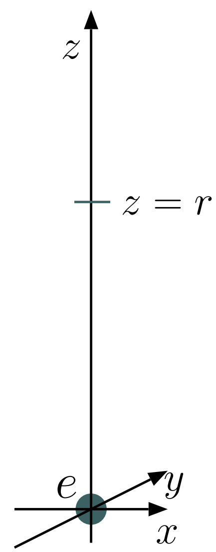
where

$$r=\sqrt{x^2+y^2+z^2} \text{ and } \hat{r}=\frac{(x,y,z)}{r}.$$

Determine the electrostatic potential field V established by charge $Q=e$ with the provision that $V\to 0$ as $r\to\infty$ (i.e., ground at infinity).

Solution: Field $\mathbf{E}$ and its potential V will exhibit spherical symmetry in this problem. Therefore, with no loss of generality, we can calculate the line integral from a point p at a distance r from the origin to a point o at ∞ (the specified ground) along, say, the z-axis. Approaching the problem that way, the potential drop from r to ∞ is

$$\begin{aligned}V(r) &= \int_{z=r}^{\infty}\frac{e}{4\pi\epsilon_o z^2}\hat{z}\cdot\hat{z}dz\\ &= -\frac{e}{4\pi\epsilon_o z}\Big|_r^{\infty}=\frac{e}{4\pi\epsilon_o r}.\end{aligned}$$



- To convert electrostatic potential V_p (in volts) at any point p to potential energy of a charge q brought to the same point, it is sufficient to multiply V_p with q (or just the sign of q, depending on which energy units we want to use — see the next example).

Example 6: In view of Example 5, what are the potential energies of a proton e and an electron $-e$ placed at distance $r = a$ away from the proton at the origin, where distance

$$a \equiv \frac{4\pi\epsilon_o}{e^2}\frac{\hbar^2}{m_e} = 0.529 \times 10^{-10}\,\text{m}$$

stands for *Bohr radius* — it is the mean distance of the ground state electron in a hydrogen atom from the center of the atom. Recall that $e = 1.602 \times 10^{-19}$ C and $\epsilon_o \approx 10^{-9}/36\pi$ F/m.

Solution: Let's first evaluate the potential $V(r)$ at $r = a$:

$$V(a) = \frac{e}{4\pi\epsilon_o a} \approx \frac{(1.6 \times 10^{-19})36\pi \times 10^9}{4\pi \times 0.53 \times 10^{-10}} = \frac{9 \times 1.6}{0.53} = 27.2\,\text{V}.$$

For the proton, potential energy in Joules is calculated by multiplying $V(a) = 27.2$ V with $q = e = 1.602 \times 10^{-19}$ C. However, by referring to 1.602×10^{-19} J of energy as 1 eV (electron-volt), it is more convenient to refer to potential energy $eV(a)$ of the proton at $r = a$ as

$$eV(a) = 27.2\,\text{eV}.$$

Likewise, for a particle with charge $q = -e$, i.e., an electron, potential energy at the same location is

$$-eV(a) = -27.2\,\text{eV}.$$