## 6 Circulation and boundary conditions

Since curl-free static electric fields have path-independent line integrals, it follows that over closed paths $C$ (when points $p$ and $o$ coincide)

$$
\oint_{C} \mathbf{E} \cdot d \mathbf{l}=0,
$$

where the $\oint_{C} \mathbf{E} \cdot d \mathbf{l}$ is called the circulation of field $\mathbf{E}$ over closed path $C$ bounding a surface $S$ (see margin).

Example 1: Consider the static electric field variation

$$
\mathbf{E}(x, y, z)=\hat{x} \frac{\rho x}{\epsilon_{o}}
$$

that will be encountered within a uniformly charged slab of an infinite extent in $y$ and $z$ directions and a finite width in $x$ direction centered about $x=0$. Show that this field $\mathbf{E}$ satisfies the condition $\oint_{C} \mathbf{E} \cdot d \mathbf{l}=0$ for a rectangular closed path $C$ with vertices at $(x, y, z)=(-3,0,0),(3,0,0),(3,4,0)$, and $(-3,4,0)$ traversed in the order of the vertices given.

Solution: Integration path $C$ is shown in the figure in the margin. With the help of the figure we expand the circulation $\oint_{C} \mathbf{E} \cdot d \mathbf{l}$ as

$$
\begin{aligned}
\mathbf{E} & =\int_{x=-3}^{3} \hat{x} \frac{\rho x}{\epsilon_{o}} \cdot \hat{x} d x+\int_{y=0}^{4} \hat{x} \frac{\rho 3}{\epsilon_{o}} \cdot \hat{y} d y+\int_{x=3}^{-3} \hat{x} \frac{\rho x}{\epsilon_{o}} \cdot \hat{x} d x+\int_{y=4}^{0} \hat{x} \frac{\rho(-3)}{\epsilon_{o}} \cdot \hat{y} d y \\
& =\int_{x=-3}^{3} \frac{\rho x}{\epsilon_{o}} d x+0+\int_{x=3}^{-3} \frac{\rho x}{\epsilon_{o}} d x+0=0
\end{aligned}
$$



Closed loop integral over path $C$ enclosing surface $S$.

Note that the area increment dS of surface $S$ is taken by convention to point in the right-hand-rule direction with respect to "circulation" direction $C$.


Note that in expanding $\oint_{C} \mathbf{E} \cdot d \mathbf{l}$ above for the given path $C$, we took $d \mathbf{l}$ as $\hat{x} d x$ and $\hat{y} d y$ in turns (along horizontal and vertical edges of $C$, respectively) and ordered the integration limits in $x$ and $y$ to traverse $C$ in a counter-clockwise direction as indicated in the diagram.

- Vector fields $\mathbf{E}$ having zero circulations over all closed paths $C$ are known as conservative fields (for obvious reasons having to do with their use in modeling static fields compatible with conservation theorems).
- The concepts of curl-free and conservative fields overlap, that is

$$
\oint_{C} \mathbf{E} \cdot d \mathbf{l}=0 \quad \Leftrightarrow \quad \nabla \times \mathbf{E}=0
$$

over all closed paths $C$ and at each $\mathbf{r}$.

- The above relationship between circulation and curl is also a consequence of Stoke's theorem (discussed in MATH 241) which asserts that

$$
\oint_{C} \mathbf{E} \cdot d \mathbf{l}=\int_{S} \nabla \times \mathbf{E} \cdot d \mathbf{S}
$$

where

- the integration surface $S$ on the right is bounded by the closed integration contour $C$ of the left side, and


STOKE'S THM:
Circulation of E around close path $C$ equals the flux over enclosed surface $S$ of the curl of $E$ taken in direction of dS.
dS points in right-hand-rule
direction with respect to
"circulation" direction C.

## Stoke's thm.

- the incremental area element $d \mathbf{S}$ on the right points across area $S$ in the direction indicated by a right-hand rule as follows:

Point your right thumb in chosen circulation direction $C$; then your right fingers point through surface $S$ in the direction that should be adopted for $d \mathbf{S}$.

- Given Stoke's theorem, $\oint_{C} \mathbf{E} \cdot d \mathbf{l}=0$ follows immediately for all $C$, if $\nabla \times \mathbf{E}=0$ is true over all $\mathbf{r}$.

Verification of Stoke's thm: Stoke's theorem applies to all contours $C$ of all sizes and orientations and their enclosed surfaces $S$ of any shape. For a small rectangular contour on a constant $x$ plane with sufficiently small $\Delta y$ and $\Delta z$ dimensions parallel to $y$ and $z$ axes (see figure in the margin), we have

$$
\oint_{C} \mathbf{E} \cdot d \mathbf{l} \approx\left(E_{z \mid 2}-E_{z \mid 1}\right) \Delta z-\left(E_{y \mid 4}-E_{y \mid 3}\right) \Delta y,
$$

an approximation that can also be re-arranged as

$$
\oint_{C} \mathbf{E} \cdot d \mathbf{l} \approx\left(\frac{E_{z \mid 2}-E_{z \mid 1}}{\Delta y}-\frac{E_{y \mid 4}-E_{y \mid 3}}{\Delta z}\right) \hat{x} \cdot \Delta y \Delta z \hat{x} .
$$

Right hand side above is clearly an approximation also for

$$
(\nabla \times \mathbf{E}) \cdot d \mathbf{S}=(\nabla \times \mathbf{E}) \cdot d y d z \hat{x}=\left(\frac{\partial E_{z}}{\partial y}-\frac{\partial E_{y}}{\partial z}\right) \hat{x} \cdot d y d z \hat{x}
$$

Matching the approximations and taking the limit of vanishing $\Delta y$ and $\Delta z$, we realize that for any infinitesimal area element $d \mathbf{S}$ of an arbitrary direction,

$$
\nabla \times \mathbf{E} \cdot d \mathbf{S}=\oint_{d C} \mathbf{E} \cdot d \mathbf{l}
$$

where $d C$ denotes the bounding infinitesimal contour of $d \mathbf{S}$ traversed in the right-hand rule direction. Stokes theorem for an arbitrary $C$ over a finite enclosed area $S$ is obtained by superposing these infinitesimals - the left side then becomes $\int_{S} \nabla \times \mathbf{E} \cdot d \mathbf{S}$ and the right side $\oint_{C} \mathbf{E} \cdot d \mathbf{l}$ after cancellations of opposing line integral contributions coming from overlapping adjacent segments (see figure in the margin).

- Stoke's theorem clearly implies that curl is circulation per unit area, just as the divergence theorem showed that divergence is flux per unit volume. The only difference is, curl also has a direction, which is the normal unit of the plane that contains the maximal value of circulation per unit area found at that location over all possible orientations of $d \mathbf{S}$.

We can now summarize the general constraints governing static electric fields as

$$
\nabla \times \mathbf{E}(\mathbf{r})=0, \quad \nabla \cdot \mathbf{D}(\mathbf{r})=\rho(\mathbf{r}), \quad \text { where } \mathbf{D}(\mathbf{r})=\epsilon_{o} \mathbf{E}(\mathbf{r}) .
$$

- Vector fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{D}(\mathbf{r})$ governed by these equations will in general be continuous functions of position coordinates $\mathbf{r}=(x, y, z)$ except at


Sum of circulations over small squares cancel in the interior edges and only survive around the exterior path C. This way, circulation around $C$ matches the sum of the fluxes of curl E calculated over the small squares.

## Laws of electrostatics:

$$
\begin{aligned}
\nabla \times \mathbf{E} & =0 \\
\nabla \cdot \epsilon_{o} \mathbf{E} & =\rho
\end{aligned}
$$

They also apply "quasi-statically" over a region of dimension $L$ when a time-varying field source $\rho(\mathbf{r}, t)$ has a time-constant $\tau$ much longer than the propagation time delay $L / c$ of $\mathbf{E}(\mathbf{r}, t)$ field variations across the region ( $c$ is the speed of light).

In electro-quasistatics (EQS) $\mathbf{E}(\mathbf{r}, t)$ will be accompanied by a slowly varying magnetic field $\mathbf{B}(\mathbf{r}, t)$ (to be studied starting in Lecture 12).
boundary surfaces where charge density function $\rho(\mathbf{r})$ requires a representation in terms of a surface charge density $\rho_{s}(\mathbf{r})$.

- For instance, according to our earlier results, static electric field of a charge density (see sketch at the margin)

$$
\rho(\mathbf{r})=\rho_{s} \delta(z)
$$

would be

$$
\mathbf{E}(\mathbf{r})=\hat{z} \frac{\rho_{s}}{2 \epsilon_{o}} \operatorname{sgn}(z) \Rightarrow \mathbf{D}(\mathbf{r})=\hat{z} \frac{\rho_{s}}{2} \operatorname{sgn}(z) .
$$

- Consider a superposition of these fields with fields $\mathbf{E}_{o}(\mathbf{r})$ and $\mathbf{D}_{o}(\mathbf{r})=\epsilon_{o} \mathbf{E}_{o}(\mathbf{r})$ produced by arbitrary continuous sources, namely (macroscopic) fields


$$
\mathbf{E}(\mathbf{r})=\hat{z} \frac{\rho_{s}}{2 \epsilon_{o}} \operatorname{sgn}(z)+\mathbf{E}_{o}(\mathbf{r}) \text { and } \mathbf{D}(\mathbf{r})=\hat{z} \frac{\rho_{s}}{2} \operatorname{sgn}(z)+\epsilon_{o} \mathbf{E}_{o}(\mathbf{r}) .
$$

Since fields $\mathbf{E}_{o}(\mathbf{r})$ and $\mathbf{D}_{o}(\mathbf{r})$ vary continuously, these field expressions must satisfy

$$
\hat{z} \cdot\left(\mathbf{D}^{+}-\mathbf{D}^{-}\right)=\rho_{s} \quad \text { and } \quad \hat{z} \times\left(\mathbf{E}^{+}-\mathbf{E}^{-}\right)=0
$$

where

$$
\mathbf{E}^{+} \equiv \mathbf{E}\left(x, y, 0^{+}\right) \text {and } \mathbf{E}^{-} \equiv \mathbf{E}\left(x, y, 0^{-}\right)
$$

refer to limiting values of $\mathbf{E}$ at $z=0$ plane from above and below, respectively, and likewise for


$$
\mathbf{D}^{+} \equiv \mathbf{D}\left(x, y, 0^{+}\right) \text {and } \mathbf{D}^{-} \equiv \mathbf{D}\left(x, y, 0^{-}\right)
$$

- The above "boundary condition equations" can be written in a more general form (see margin for justification) as

$$
\hat{n} \cdot\left(\mathbf{D}^{+}-\mathbf{D}^{-}\right)=\rho_{s} \quad \text { and } \quad \hat{n} \times\left(\mathbf{E}^{+}-\mathbf{E}^{-}\right)=0
$$

where $\hat{n}$ denotes a unit vector normal to any surface of an arbitrary orientation carrying a surface charge density $\rho_{s}$, while field vectors with superscripts + and - indicate limiting values of fields measured on either side of the charged surface (with $\hat{n}$ pointing from - to + ).

- The equations can be further simplified as

$$
D_{n}^{+}-D_{n}^{-}=\rho_{s} \quad \text { and } \quad E_{t}^{+}=E_{t}^{-}
$$

where $D_{n}$ and $E_{t}$ refer to normal component of $\mathbf{D}$ and tangential component of $\mathbf{E}$, respectively. Clearly, these boundary conditions say that at any surface $S$,

- tangential component of electric field $\mathbf{E}$ needs to be continuous, but
- normal component of $\mathbf{D}$ can change by an amount equal to the charge density $\rho_{s}$ carried by the surface.


Constraint

$$
\oint_{C} \mathbf{E} \cdot d \mathbf{l}=0
$$

around the dotted path yields

$$
E_{t}^{+}=E_{t}^{-}
$$

in $w \rightarrow 0$ limit.

Gauss's law

$$
\oint_{S} \mathbf{D} \cdot d \mathbf{S}=Q_{V}
$$

applied over the dotted volume (seen in profile) yields

$$
D_{n}^{+}-D_{n}^{-}=\rho_{s}
$$

in $w \rightarrow 0$ limit.

$$
\mathbf{D}=0 \text { for } x<0 .
$$

## Example 2:

Measurements indicate that $\mathbf{D}=0$ in the region $x<0$.
Also, $x=0$ and $x=5 \mathrm{~m}$ planes contain surface charge densities of $\rho_{s}=2 \mathrm{C} / \mathrm{m}^{2}$ and $\rho_{s o}$, respectively.

Determine $\rho_{\text {so }}$ and $\mathbf{D}$ for $-\infty<x<\infty$ if there are no other charge distributions.

## Solution:

Since the normal component of $\mathbf{D}$ must increase by $\rho_{s}=2 \mathrm{C} / \mathrm{m}^{2}$ when we cross the charged surface $x=0$, we must have $\mathbf{D}=\hat{x} 2 \mathrm{C} / \mathrm{m}^{2}$ in the region $0<x<5 \mathrm{~m}$.

Having $\mathbf{D}=0$ in the region $x<0$ requires that the field due to surface charge $\rho_{s o}$ on $x=5 \mathrm{~m}$ plane must cancel the field due $\rho_{s}=2 \mathrm{C} / \mathrm{m}^{2}$ on $x=0$ plane - this requires that $\rho_{s o}$ be $-2 \mathrm{C} / \mathrm{m}^{2}$.

In that case $\mathbf{D}=0$ in the region $x>5 \mathrm{~m}$, because $\mathbf{D}$ must increase by $\rho_{s o}=-2$ $\mathrm{C} / \mathrm{m}^{2}$ when we cross the charged surface at $x=5 \mathrm{~m}$.

$$
\mathbf{D}=3 \hat{y} \text { for } x<0 .
$$

Example 3: In the region $x<0$ measurements indicate a constant displacement field $\mathbf{D}=3 \hat{y} \mathrm{C} / \mathrm{m}^{2}$. Also, $x=0$ and $x=5 \mathrm{~m}$ planes contain surface charge densities of $\rho_{s}=2 \mathrm{C} / \mathrm{m}^{2}$ and $\rho_{s}=-6 \mathrm{C} / \mathrm{m}^{2}$ respectively. Determine $\mathbf{D}$ for $x>0$ if $\mathbf{D}$ is known to be uniform in the intervals $0<x<5 \mathrm{~m}$ and $x>5 \mathrm{~m}$.

Solution: First we note that $\mathbf{E}=\frac{\mathrm{D}}{\epsilon_{o}}=\hat{y} \frac{3}{\epsilon_{o}} \mathrm{~V} / \mathrm{m}$ is tangential to $x=0$ and $x=5 \mathrm{~m}$ surfaces. Since the tangential component of $\mathbf{E}$ cannot change at any boundary, we will have a uniform $E_{y}=\frac{3}{\epsilon_{o}}$ in all regions, $-\infty<x<\infty$, implying that $D_{y}=3 \mathrm{C} / \mathrm{m}^{2}$ throughout (caused by charges at $|y| \rightarrow \infty$ ).

Second, we note that normal component of $\mathbf{D}$ with respect to $x=0$ and $x=5 \mathrm{~m}$ surfaces, namely $D_{x}$, is zero in $z<0$. Since the normal component of $\mathbf{D}$ must increase by an amount $\rho_{s}$ when we cross a charged surface, we must have $D_{x}=2$ $\mathrm{C} / \mathrm{m}^{2}$ in the region $0<x<5 \mathrm{~m}$, and $D_{x}=2+(-6)=-4 \mathrm{C} / \mathrm{m}^{2}$ in $x>5 \mathrm{~m}$.

In summary,

$$
\mathbf{D}= \begin{cases}\hat{y} 3, & \text { for } x<0, \\ \hat{x} 2+\hat{y} 3, & \text { for } 0<x<5 \mathrm{~m} \frac{\mathrm{C}}{\mathrm{~m}^{2}} \\ -\hat{x} 4+\hat{y} 3, & \text { for } x>5 \mathrm{~m}\end{cases}
$$

