6 Circulation and boundary conditions

Since curl-free static electric fields have path-independent line integrals, it follows that over closed paths C (when points p and o coincide)

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0,$$

where the $\oint_C \mathbf{E} \cdot d\mathbf{l}$ is called the **circulation** of field **E** over closed path *C* bounding a surface *S* (see margin).

Example 1: Consider the static electric field variation

$$\mathbf{E}(x, y, z) = \hat{x} \frac{\rho x}{\epsilon_o}$$

that will be encountered within a uniformly charged slab of an infinite extent in y and z directions and a finite width in x direction centered about x = 0. Show that this field **E** satisfies the condition $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$ for a rectangular closed path C with vertices at (x, y, z) = (-3, 0, 0), (3, 0, 0), (3, 4, 0), and (-3, 4, 0) traversed in the order of the vertices given.

Solution: Integration path C is shown in the figure in the margin. With the help of the figure we expand the circulation $\oint_C \mathbf{E} \cdot d\mathbf{l}$ as

$$\mathbf{E} = \int_{x=-3}^{3} \hat{x} \frac{\rho x}{\epsilon_o} \cdot \hat{x} dx + \int_{y=0}^{4} \hat{x} \frac{\rho 3}{\epsilon_o} \cdot \hat{y} dy + \int_{x=3}^{-3} \hat{x} \frac{\rho x}{\epsilon_o} \cdot \hat{x} dx + \int_{y=4}^{0} \hat{x} \frac{\rho(-3)}{\epsilon_o} \cdot \hat{y} dy$$
$$= \int_{x=-3}^{3} \frac{\rho x}{\epsilon_o} dx + 0 + \int_{x=3}^{-3} \frac{\rho x}{\epsilon_o} dx + 0 = 0.$$



Closed loop integral over path C enclosing surface S.

Note that the area increment dS of surface S is taken by convention to point in the right-hand-rule direction with respect to "circulation" direction C.



Note that in expanding $\oint_C \mathbf{E} \cdot d\mathbf{l}$ above for the given path C, we took $d\mathbf{l}$ as $\hat{x}dx$ and $\hat{y}dy$ in turns (along horizontal and vertical edges of C, respectively) and ordered the integration limits in x and y to traverse C in a counter-clockwise direction as indicated in the diagram.

- Vector fields **E** having zero circulations over all closed paths *C* are known as **conservative fields** (for obvious reasons having to do with their use in modeling static fields compatible with conservation theorems).
 - The concepts of *curl-free* and *conservative* fields overlap, that is

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \quad \Leftrightarrow \quad \nabla \times \mathbf{E} = 0$$

over all closed paths C and at each \mathbf{r} .

• The above relationship between circulation and curl is also a consequence of **Stoke's theorem** (discussed in MATH 241) which asserts that **Ste**

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S},$$

where

- the integration surface S on the right is bounded by the closed integration contour C of the left side, and



STOKE'S THM: Circulation of E around close path C equals the flux over enclosed surface S of the curl of E taken in direction of dS.

dS points in right-hand-rule direction with respect to "circulation" direction C.

Stoke's thm.

- the incremental area element $d\mathbf{S}$ on the right points across area S_{-z} in the direction indicated by a **right-hand rule** as follows:

Point your right thumb in chosen circulation direction C; then your right fingers point through surface S in the direction that should be adopted for $d\mathbf{S}$.

- Given Stoke's theorem, $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$ follows immediately for all C, if $\nabla \times \mathbf{E} = 0$ is true over all \mathbf{r} .

Verification of Stoke's thm: Stoke's theorem applies to all contours C of all sizes and orientations and their enclosed surfaces S of any shape. For a small rectangular contour on a constant x plane with sufficiently small Δy and Δz dimensions parallel to y and z axes (see figure in the margin), we have

$$\oint_C \mathbf{E} \cdot d\mathbf{l} \approx (E_{z|2} - E_{z|1}) \Delta z - (E_{y|4} - E_{y|3}) \Delta y,$$

an approximation that can also be re-arranged as

$$\oint_C \mathbf{E} \cdot d\mathbf{l} \approx \left(\frac{E_{z|2} - E_{z|1}}{\Delta y} - \frac{E_{y|4} - E_{y|3}}{\Delta z}\right) \hat{x} \cdot \Delta y \Delta z \, \hat{x}.$$

Right hand side above is clearly an approximation also for

$$(\nabla \times \mathbf{E}) \cdot d\mathbf{S} = (\nabla \times \mathbf{E}) \cdot dy dz \, \hat{x} = (\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) \hat{x} \cdot dy dz \hat{x}.$$



STOKE'S THM: Circulation of E around close path C equals the flux over enclosed surface S of the curl of E taken in direction of dS.

dS points in right-hand-rule direction with respect to "circulation" direction C.



Matching the approximations and taking the limit of vanishing Δy and Δz , we realize that for any infinitesimal area element $d\mathbf{S}$ of an arbitrary direction,

$$\nabla \times \mathbf{E} \cdot d\mathbf{S} = \oint_{dC} \mathbf{E} \cdot d\mathbf{I}$$

where dC denotes the bounding infinitesimal contour of $d\mathbf{S}$ traversed in the right-hand rule direction. Stokes theorem for an arbitrary C over a finite enclosed area S is obtained by superposing these infinitesimals — the left side then becomes $\int_{S} \nabla \times \mathbf{E} \cdot d\mathbf{S}$ and the right side $\oint_{C} \mathbf{E} \cdot d\mathbf{l}$ after cancellations of opposing line integral contributions coming from overlapping adjacent segments (see figure in the margin).

- Stoke's theorem clearly implies that **curl is circulation per unit area**, just as the divergence theorem showed that **divergence is flux per unit volume**. The only difference is, curl also has a direction, which is the normal unit of the plane that contains the maximal value of circulation per unit area found at that location over all possible orientations of $d\mathbf{S}$.

We can now summarize the general constraints governing static electric fields as

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0, \quad \nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}), \text{ where } \mathbf{D}(\mathbf{r}) = \epsilon_o \mathbf{E}(\mathbf{r}).$$

• Vector fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{D}(\mathbf{r})$ governed by these equations will in general be continuous functions of position coordinates $\mathbf{r} = (x, y, z)$ except at



Sum of circulations over small squares cancel in the interior edges and only survive around the exterior path C. This way, circulation around C matches the sum of the fluxes of curl E calculated over the small squares.

Laws of electrostatics:

 $\nabla \times \mathbf{E} = 0$ $\nabla \cdot \epsilon_o \mathbf{E} = \rho$

They also apply "quasi-statically" over a region of dimension Lwhen a time-varying field source $\rho(\mathbf{r}, t)$ has a *time-constant* τ much longer than the propagation time delay L/c of $\mathbf{E}(\mathbf{r}, t)$ field variations across the region (c is the speed of light).

In electro-quasistatics (EQS) $\mathbf{E}(\mathbf{r}, t)$ will be accompanied by a slowly varying magnetic field $\mathbf{B}(\mathbf{r}, t)$ (to be studied starting in Lecture 12). boundary surfaces where charge density function $\rho(\mathbf{r})$ requires a representation in terms of a surface charge density $\rho_s(\mathbf{r})$.

- For instance, according to our earlier results, static electric field of a charge density (see sketch at the margin)

$$\rho(\mathbf{r}) =
ho_s \delta(z)$$

would be

$$\mathbf{E}(\mathbf{r}) = \hat{z} \frac{\rho_s}{2\epsilon_o} \operatorname{sgn}(z) \quad \Rightarrow \quad \mathbf{D}(\mathbf{r}) = \hat{z} \frac{\rho_s}{2} \operatorname{sgn}(z).$$

• Consider a superposition of these fields with fields $\mathbf{E}_o(\mathbf{r})$ and $\mathbf{D}_o(\mathbf{r}) = \epsilon_o \mathbf{E}_o(\mathbf{r})$ produced by arbitrary continuous sources, namely (macroscopic) fields

$$\mathbf{E}(\mathbf{r}) = \hat{z} \frac{\rho_s}{2\epsilon_o} \operatorname{sgn}(z) + \mathbf{E}_o(\mathbf{r}) \text{ and } \mathbf{D}(\mathbf{r}) = \hat{z} \frac{\rho_s}{2} \operatorname{sgn}(z) + \epsilon_o \mathbf{E}_o(\mathbf{r}).$$

Since fields $\mathbf{E}_o(\mathbf{r})$ and $\mathbf{D}_o(\mathbf{r})$ vary continuously, these field expressions must satisfy

$$\hat{z} \cdot (\mathbf{D}^+ - \mathbf{D}^-) = \rho_s \text{ and } \hat{z} \times (\mathbf{E}^+ - \mathbf{E}^-) = 0$$

where

$$\mathbf{E}^+ \equiv \mathbf{E}(x, y, 0^+)$$
 and $\mathbf{E}^- \equiv \mathbf{E}(x, y, 0^-)$

refer to limiting values of \mathbf{E} at z = 0 plane from *above* and *below*, respectively, and likewise for

$$\mathbf{D}^+ \equiv \mathbf{D}(x,y,0^+) \text{ and } \mathbf{D}^- \equiv \mathbf{D}(x,y,0^-).$$

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• The above "boundary condition equations" can be written in a more general form (see margin for justification) as

$$\hat{n} \cdot (\mathbf{D}^+ - \mathbf{D}^-) = \rho_s \text{ and } \hat{n} \times (\mathbf{E}^+ - \mathbf{E}^-) = 0$$

where \hat{n} denotes a unit vector normal to any surface of an arbitrary orientation carrying a surface charge density ρ_s , while field vectors with superscripts + and - indicate limiting values of fields measured on either side of the charged surface (with \hat{n} pointing from - to +).

- The equations can be further simplified as

$$D_n^+ - D_n^- = \rho_s$$
 and $E_t^+ = E_t^-$

where D_n and E_t refer to normal component of **D** and tangential component of **E**, respectively. Clearly, these **boundary conditions** say that at any surface S,

- \circ tangential component of electric field ${\bf E}$ needs to be continuous, but
- normal component of **D** can change by an amount equal to the charge density ρ_s carried by the surface.



Constraint

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

around the dotted path yields

$$E_t^+ = E_t^-$$

in $w \to 0$ limit.

Gauss's law

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_V$$

applied over the dotted volume (seen in profile) yields

$$D_n^+ - D_n^- = \rho_s$$

in $w \to 0$ limit.

Example 2:

Measurements indicate that $\mathbf{D} = 0$ in the region x < 0.

Also, x = 0 and x = 5 m planes contain surface charge densities of $\rho_s = 2$ C/m² and ρ_{so} , respectively.

Determine ρ_{so} and **D** for $-\infty < x < \infty$ if there are no other charge distributions.

Solution:

- Since the normal component of **D** must increase by $\rho_s = 2 \text{ C/m}^2$ when we cross the charged surface x = 0, we must have $\mathbf{D} = \hat{x} 2 \text{ C/m}^2$ in the region 0 < x < 5 m.
- Having $\mathbf{D} = 0$ in the region x < 0 requires that the field due to surface charge ρ_{so} on x = 5 m plane must cancel the field due $\rho_s = 2 \text{ C/m}^2$ on x = 0 plane this requires that ρ_{so} be -2 C/m^2 .

In that case $\mathbf{D} = 0$ in the region x > 5 m, because \mathbf{D} must increase by $\rho_{so} = -2$ C/m² when we cross the charged surface at x = 5 m.



- **Example 3:** In the region x < 0 measurements indicate a constant displacement field $\mathbf{D} = 3\hat{y} \text{ C/m}^2$. Also, x = 0 and x = 5 m planes contain surface charge densities of $\rho_s = 2 \text{ C/m}^2$ and $\rho_s = -6 \text{ C/m}^2$ respectively. Determine \mathbf{D} for x > 0 if \mathbf{D} is known to be uniform in the intervals 0 < x < 5 m and x > 5 m.
- **Solution:** First we note that $\mathbf{E} = \frac{\mathbf{D}}{\epsilon_o} = \hat{y}\frac{3}{\epsilon_o}$ V/m is tangential to x = 0 and x = 5 m surfaces. Since the tangential component of \mathbf{E} cannot change at any boundary, we will have a uniform $E_y = \frac{3}{\epsilon_o}$ in all regions, $-\infty < x < \infty$, implying that $D_y = 3 \text{ C/m}^2$ throughout (caused by charges at $|y| \to \infty$).
 - Second, we note that normal component of **D** with respect to x = 0 and x = 5 m surfaces, namely D_x , is zero in z < 0. Since the normal component of **D** must increase by an amount ρ_s when we cross a charged surface, we must have $D_x = 2$ C/m² in the region 0 < x < 5 m, and $D_x = 2 + (-6) = -4$ C/m² in x > 5 m.

In summary,

$$\mathbf{D} = \begin{cases} \hat{y}3, & \text{for } x < 0, \\ \hat{x}2 + \hat{y}3, & \text{for } 0 < x < 5 \text{ m} \frac{\text{C}}{\text{m}^2}. \\ -\hat{x}4 + \hat{y}3, & \text{for } x > 5 \text{ m} \end{cases}$$

