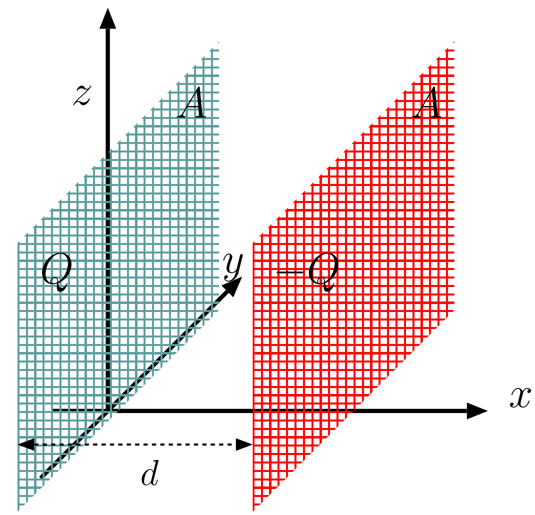
10 Capacitance and conductance



Parallel-plate capacitor: Consider a pair of conducting plates with surface areas A separated by some distance d in free space (see margin).

The plates are initially charge neutral, but then some amount of electrons are transferred from one plate to the other so that the plates acquire equal and opposite charges Q and $-Q$, distributed with surface densities of $\pm\frac{Q}{A}$ on plate surfaces facing one another (as shown in the margin).

- That way, in steady state and for $d \ll \sqrt{A}$, a field configuration confined mainly to the region between the plates is acquired, satisfying the condition that static field inside a conductor should be zero. A weak "fringing field" can be ignored if $d \ll \sqrt{A}$ and thus the geometry well approximates the case with infinite plates.

  – A constant displacement field

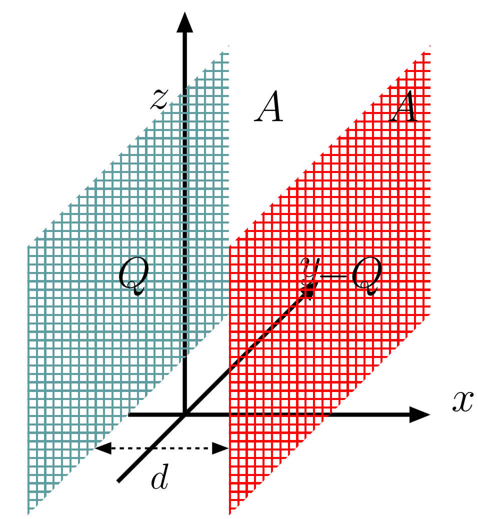
  $$\mathbf{D} = \hat{x}\frac{Q}{A}$$

  satisfies the normal boundary condition at the left plate boundary as well as Gauss's law $\nabla \cdot \mathbf{D} = 0$ in the region between the plates. The corresponding electrostatic field is

  $$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_o} = \hat{x}\frac{Q}{\epsilon_o A},$$

  and the voltage drop from (positive charged) left plate to (negative charged) right plate is

$$V = \int_{(0,0,0)}^{(d,0,0)} \mathbf{E} \cdot d\mathbf{l} = \int_{x=0}^{d} \frac{Q}{\epsilon_o A} dx = \frac{d}{\epsilon_o A} Q.$$



The last result can be expressed as a *linear charge-voltage relation*

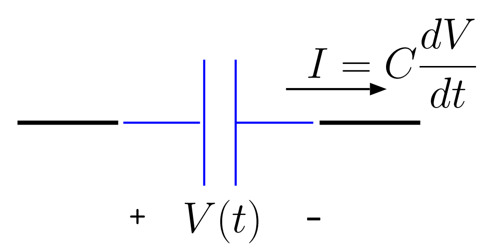
$$Q = CV$$

with

$$C \equiv \epsilon_o \frac{A}{d}$$

representing the **capacitance** of the parallel conducting plate arrangement that we call **parallel plate capacitor**.

- As we know from our circuit courses such a capacitor can be used in diverse ways in time-varying filter circuits as well as for energy and charge storage. A capacitor connected to an external circuit will conduct a current $I = \frac{dQ}{dt}$ flowing externally in the direction of voltage drop V across capacitor plates, obeying a *linear current-voltage relation*



$$I = C\frac{dV}{dt}.$$

This IV-relation follows from the QV-relation above *quasi-statically* assuming that $\sqrt{A} \ll \lambda = c/f$, where f is the highest frequency of $V(t)$. The power absorbed by the capacitor is then calculated as

$$P = VI = VC\frac{dV}{dt} = \frac{d}{dt}(\frac{1}{2}CV^2),$$

implying a stored energy of

$$W = \frac{1}{2}CV^2 = \frac{1}{2}\epsilon_o|E_x|^2Ad$$

when the capacitor is in a charged state.

- Notice that stored energy is

$$\frac{1}{2}\epsilon_o E_x^2 = \frac{1}{2}\epsilon_o \mathbf{E}\cdot\mathbf{E}$$

times the volume Ad occupied by the field $\mathbf{E}$ between the capacitor plates. That suggests that

$$w = \frac{1}{2}\epsilon_o \mathbf{E}\cdot\mathbf{E}$$

can be interpreted as stored electrostatic energy per unit volume in general.

  – Also both capacitance C and stored energies W and w would have ϵ replacing ϵ_o in dielectric media.

A capacitor with a perfect dielectric between its plates will hold its charge and stored energy indefinitely. However, if the dielectric is imperfect and has a finite **conductivity** σ, charge will be transported from the positive to negative plate by a volumetric current density

$$\mathbf{J} = \sigma\mathbf{E},$$

which will result in a *quasi-static* discharge of the capacitor and the loss of the stored energy W to Ohmic dissipation in the imperfect dielectric.

Just as **capacitance** C characterizes the energy and charge storage "capacity" of the capacitor, we can define a **conductance** G that relates the quasi-static discharge current I to potential drop V:

- Discharge current I is the product of current density

$$J_x = \sigma E_x$$

  in A/m^2 units and the plate area A. Since $E_x = \frac{V}{d}$, we obtain a *linear current-voltage relation*

$$I = GV$$

  with conductance

$$G \equiv \sigma \frac{A}{d}$$

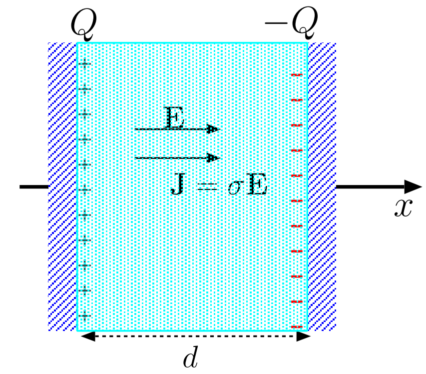
  for the parallel plate capacitor.

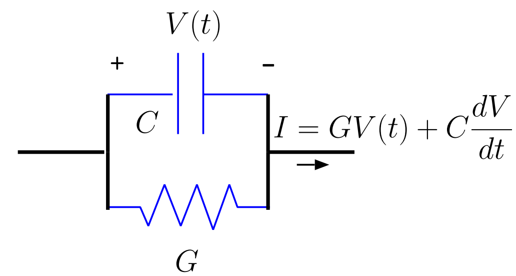
  – Notice that $G = \frac{\sigma}{\epsilon} C$, a relation that will hold true for other types of capacitors that we will be examining.

    ◦ Also,

$$R \equiv \frac{1}{G} = \frac{d}{A\sigma}$$

    is the corresponding **resistance** that scales inversely with conductivity σ of the material — large σ materials will have small resistance, but for a given σ, R increases with length d and





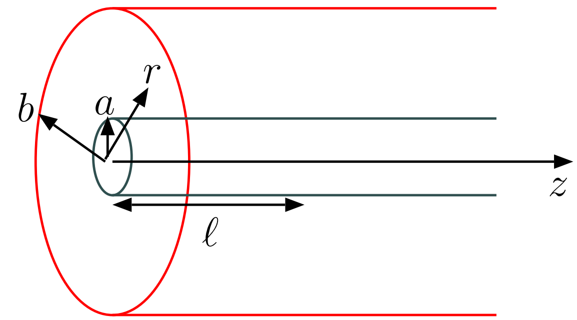
System above behaves like a resistor $R = 1/G$ for

$$\omega \ll \frac{G}{C} = \frac{1}{RC} = \frac{\sigma}{\epsilon}$$

and a capacitor C in the complementary frequency band. To obtain capacitor behavior at low frequencies make sure that σ is sufficiently small.

Alternatively, with large σ the system becomes a good electrical connector, a resistor R with a small resistance $R \propto 1/\sigma$.

decreases with increasing cross-sectional area A. Simple conductivity models and $\mathbf{J}$ will be discussed next lecture.



Coaxial Cable: When we study guided wave propagation later in the course we will learn about coaxial cables.

- A coax cable consists of two conducting regions — a central cylindrical conductor with a cross-sectional radius a, enclosed by a conducting pipe of a radius $b > a$ (see margin), with some dielectric ϵ filling in the space. We will next calculate the capacitance and conductance of a coax segment of some length ℓ.

- For $\ell \gg b$, field $\mathbf{E}$ can be assumed to point out radially away from the inner conductor with radius a to the outer conductor with radius b. In that case Gauss's law in integral form can be utilized to determine the radial field E_r. Considering a cylindrical integration surface with a radius $r > a$ centered about the inner conductor, we re-write Gauss's law

$$\epsilon \oint_s \mathbf{E} \cdot d\mathbf{S} = Q_V$$

as

$$\epsilon E_r 2\pi r \ell = Q$$

where Q is the total charge distributed over the inner conductor and ϵ the permittivity of the dielectric separating the two conductors.

– It follows that

$$E_r = \frac{Q}{2\pi\epsilon\ell r},$$

and voltage drop from inner to outer conductor is

$$V = \int_{r=a}^{b} E_r dr = \int_{r=a}^{b} \frac{Q}{2\pi\epsilon\ell r} dr = \frac{Q}{2\pi\ell\epsilon} \int_{r=a}^{b} \frac{dr}{r} = \frac{Q}{2\pi\ell\epsilon} \ln\frac{b}{a}.$$

Clearly, once again $Q = CV$, with

$$C = \frac{2\pi}{\ln\frac{b}{a}}\ell\epsilon$$

representing the capacitance of the coax segment of length l.

- The **capacitance** of the coax **per unit length** is

$$\mathcal{C} = \frac{2\pi}{\ln\frac{b}{a}}\epsilon.$$

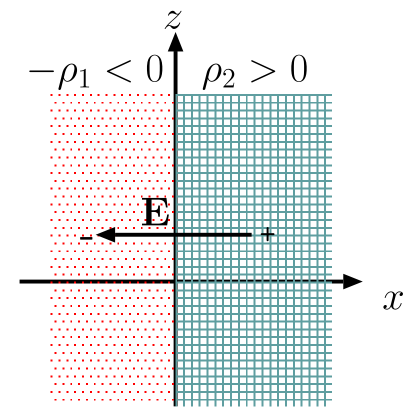
  – **Conductance** of the coax **per unit length** can likewise be shown to be

$$\mathcal{G} = \frac{2\pi}{\ln\frac{b}{a}}\sigma.$$

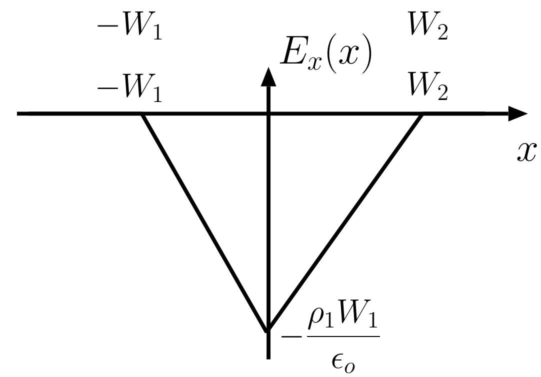
  This result is a consequence of the general relation $G = \frac{\sigma}{\epsilon}C$ mentioned earlier.

  – Per length parameters $\mathcal{C}$ and $\mathcal{G}$ of the coax will play an important role when we study guided wave propagation in coaxial transmission lines with lengths for which quasi-static approximation may be violated.

Diode junctions: In Example 4 in Lecture 7 we derived the expression for potential drop V across a charged region of a total width of $W_1 + W_2$, such that in region 1 where $-W_1 < x < 0$ the charge density $\rho = -\rho_1$ is negative, while in region 2 where $0 < x < W_2$ the charge density $\rho = \rho_2$ is positive, with the additional constraint that the entire region is charge neutral, meaning that $\rho_1 W_1 = \rho_2 W_2$.



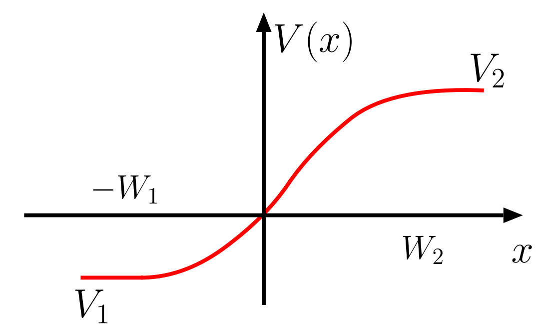
By solving *Poisson's equation* for this charge density configuration (see margin) encountered in **junction** regions of semiconductor diodes (described in detail ECE 440) we had established that the voltage drop from $x = W_2$ to $x = -W_1$ across the junction is given by



$$V = \frac{\rho_2 W_2 (W_1 + W_2)}{2\epsilon_o} = \frac{\rho_1 W_1 (W_1 + W_2)}{2\epsilon_o}.$$

The above equation implies that

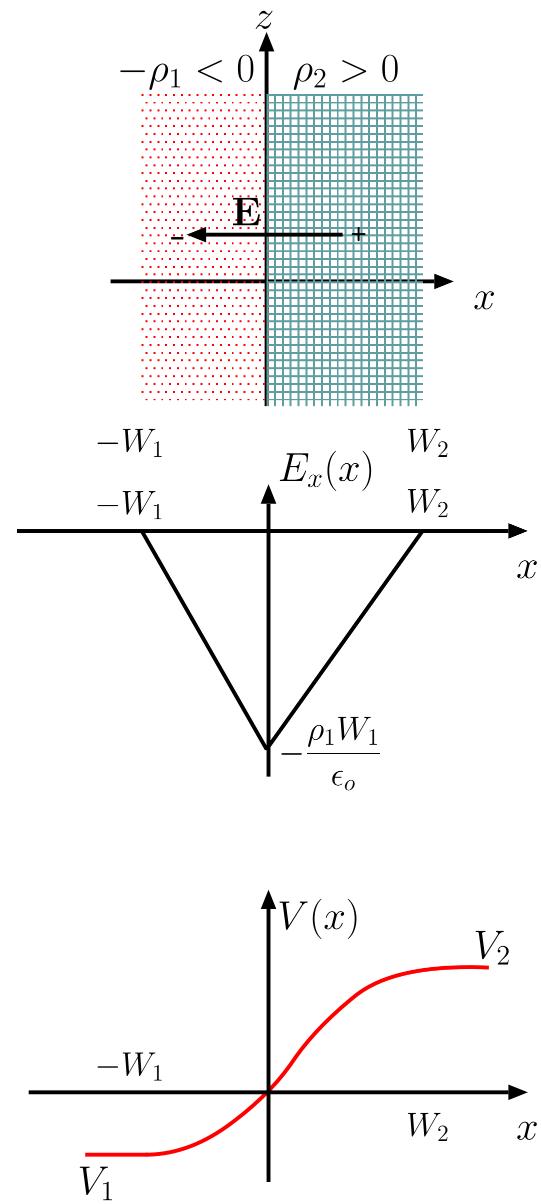
$$W_1 = \frac{2\epsilon_o V}{(W_1 + W_2)\rho_1} \text{ and } W_2 = \frac{2\epsilon_o V}{(W_1 + W_2)\rho_2} \quad \Rightarrow \quad W_1 + W_2 = \sqrt{2\epsilon_o V \frac{\rho_1 + \rho_2}{\rho_1 \rho_2}}.$$



Using the expressions above for junction voltage V and width $W_1 + W_2$, we will next derive an expression for **small signal capacitance** of the diode junction:

- In region 2 where $x > 0$, the junction holds a total positive charge of $Q = \rho_2 W_2 A$ per cross-sectional area A.

- Therefore, substituting $\frac{Q}{A}$ for $\rho_2 W_2$ in the expression for V above, and



also using the $W_1 + W_2 \propto \sqrt{V}$ expression derived above, we obtain

$$V = \frac{\rho_2 W_2 (W_1 + W_2)}{2\epsilon_o} = \frac{Q\sqrt{2\epsilon_o V \frac{\rho_1+\rho_2}{\rho_1\rho_2}}}{2\epsilon_o A},$$

which can be re-arranged as

$$Q = A\sqrt{\frac{2\epsilon_o \rho_1 \rho_2}{\rho_1 + \rho_2}}\sqrt{V}$$

representing a *non-linear* charge-voltage relation (for a given charge profiles satisfying $\rho_1 W_1 = \rho_2 W_2$).

– In a linear charge-voltage relation $Q = CV$, the capacitance parameter C represents the slope $\frac{Q}{V}$ of a Q vs V curve.

The slope of *any* Q vs V curve is given by the derivative $\frac{dQ}{dV}$, whether or not the curve is linear. The slope $\frac{dQ}{dV}$ of a *non-linear* charge-voltage curve can be interpreted as a *small signal capacitance* C. For a diode junction, differentiating the above equation, we find that

$$C = \frac{dQ}{dV} = A\sqrt{\frac{\epsilon_o}{2V}\frac{\rho_1\rho_2}{(\rho_1 + \rho_2)}}.$$

Small changes dV in junction voltage will accompany small changes $dQ = CdV$ in stored charge Q of the junction, but the amount CdV will itself depend on V because $C \propto V^{-1/2}$.