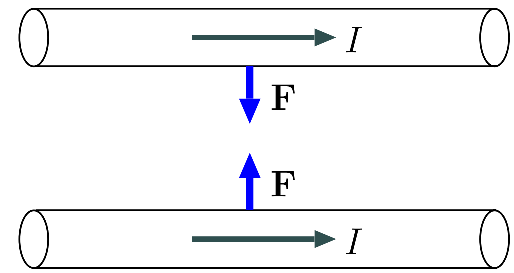
12 Magnetic force and fields and Ampere's law

Pairs of wires carrying currents I running in the same (opposite) direction are known to attract (repel) one another. In this lecture we will explain the mechanism — the phenomenon is a relativistic[1] consequence of electrostatic charge interactions, but it is more commonly described in terms of magnetic fields. This will be our introduction to magnetic field effects in this course.



"Things should be made as simple as possible – but no simpler."

— ***Albert Einstein***

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[1]**Brief summary of *special* relativity:** Observations indicate that light (EM) waves *can* be "counted" like particles and yet *travel* at one and the same speed $c = 3 \times 10^8$ m/s in *all* reference frames in relative motion. As first recognized by Albert Einstein, these facts preclude the possibility that a *particle* velocity u could appear as

$$u' = u - v \quad \text{(Newtonian)}$$

to an observer approaching the particle with a velocity v; instead, u must transform to the observer's frame as

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}, \quad \text{(relativistic)}$$

so that if $u = c$, then $u' = c$ also. This "relativistic" velocity transformation in turn requires that positions x and times t of physical events transform (between the frames) as

$$x' = \gamma(x - vt) \text{ and } t' = \gamma(t - \frac{v}{c^2}x), \quad \text{(relativistic)}$$

where $\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}}$, rather than as

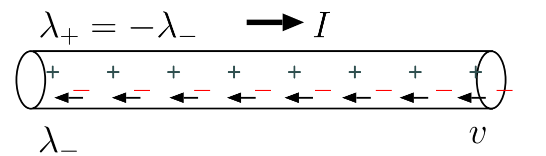
$$x' = x - vt \text{ and } t' = t, \quad \text{(Newtonian)}$$

so that $\frac{dx}{dt} = u$ and $\frac{dx'}{dt'} = u'$ are related by the relativistic formula for u' given above.

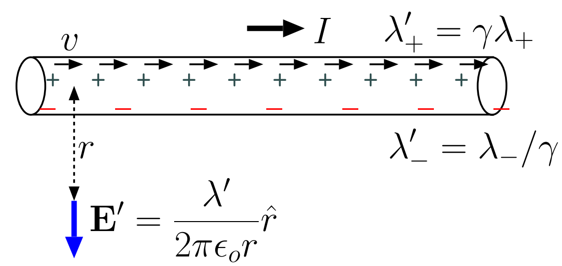
Relativistic transformations imply a number of "counter-intuitive" effects ordinarily not noticed unless $|v|$ is very close to c. One of them is *Lorentz contraction*, implied by $dx = dx'/\gamma$ at a fixed t: since $\gamma > 1$, $dx < dx'$, and moving objects having velocities v appear shorter then they are when viewed from other reference frames where v is determined. A second one is *time dilation*, implied by $dt' = dt/\gamma$ at a fixed x': since $\gamma > 1$, $dt' < dt$, and moving clocks having velocities v and fixed x' run slower than clocks in other reference frames where v is determined. Consider taking PHYS 325 to learn more about special relativity.

- Consider a current carrying stationary wire in the lab frame:
  - the wire has a stationary lattice of positive ions,
  - electrons are moving to the left through the lattice with an average speed v, and
  - a current $I > 0$ is flowing to the right as shown in the figure.
    - If the wire is electrically uncharged — which will be true if electron and ion charge densities in the wire, $\lambda_- < 0$ and $\lambda_+ > 0$, respectively, have equal magnitudes — then the wire will produce no electrostatic field $\mathbf{E}$, and any stationary charge q placed near the wire will not be subject to any force[2].
    - The current carried by the wire is $I = v|\lambda_-| = v\lambda_+$ in terms of the magnitudes of electron velocity and charge density.
- An uncharged wire in the lab frame appears as "charged" in the reference frame of the electrons carrying the current:
  - this is a *relativistic effect* due to "Lorentz contraction" of the distances between the charges in the wire.

(a) Neutral wire carrying current I in the "lab frame":



(b) In the "electron frame" the wire appears positively charged:



$$\lambda' \approx \lambda_+ \frac{v^2}{c^2} = \frac{Iv}{c^2} = Iv\mu_o\epsilon_o$$

[2]This is true for zero-resistivity wires. Current carrying wires with *finite* resistivity will however support *surface* charge densities with axial gradients to produce the static field within the wire needed to drive the current — e.g., in *Am. J. Phys.*: Jefimenko, **30**, 19 (1962); Parker, **38**, 720 (1970); Preyer, **68**, 1002 (2000).

– In the electron frame the wire is found to have a positive charge density λ', and thus it has a radial electrostatic field

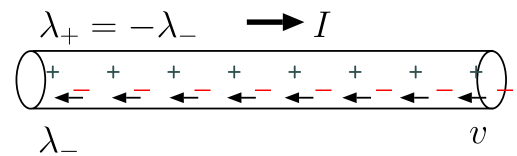
$$\mathbf{E}' = \frac{\lambda'}{2\pi\epsilon_o r}\hat{r}$$

implying an electrostatic force $\mathbf{F}' = q\mathbf{E}'$ on a stationary charge q.

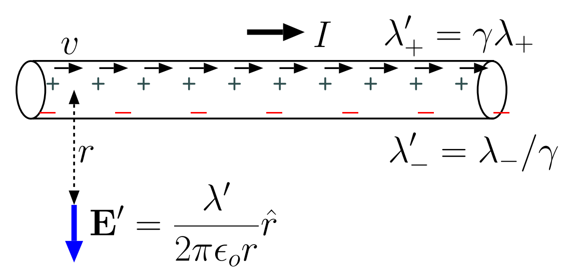
– Relativistic calculations[3] show that

$$\lambda' \approx \lambda_+ \frac{v^2}{c^2} = (\frac{I}{v})\frac{v^2}{c^2} = Iv\epsilon_o\mu_o$$

(a) Neutral wire carrying current I in the "lab frame":



(b) In the "electron frame" the wire appears positively charged:



$$\lambda' \approx \lambda_+ \frac{v^2}{c^2} = \frac{Iv}{c^2} = Iv\mu_o\epsilon_o$$

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[3](i) Electron spacings dx' measured in the electron reference frame will appear as

$$dx = \sqrt{1 - \frac{v^2}{c^2}}dx'$$

in the lab frame because of Lorentz contraction. Charge density of the electrons in the lab frame,

$$\lambda_- = \frac{\lambda'_-}{\sqrt{1 - v^2/c^2}},$$

is therefore greater in magnitude than the electron charge density λ'_- in the electron frame. Furthermore, $\lambda_- = -\lambda_+$ in order to maintain a charge neutral wire in the lab frame. (ii) Once again because of Lorentz contraction, the charge density of positive ions will appear in the electron frame as

$$\lambda'_+ = \frac{\lambda_+}{\sqrt{1 - v^2/c^2}}.$$

(iii) Thus, the total charge density of the wire in the electron frame is

$$\lambda' = \lambda'_+ + \lambda'_- = \frac{\lambda_+}{\sqrt{1 - v^2/c^2}} + \lambda_-\sqrt{1 - v^2/c^2} = \frac{\lambda_+}{\sqrt{1 - v^2/c^2}} - \lambda_+\sqrt{1 - v^2/c^2} = \frac{\lambda_+ v^2/c^2}{\sqrt{1 - v^2/c^2}} \approx \frac{\lambda_+ v^2}{c^2},$$

a *positive* density for non-zero $|v| \ll c$. (e.g., articles in *Am. J. Phys.*: Webster, **29**, 262, 1961; Matzek and Russel, **36**, 905, 1968; Arista and Lopez, **43**, 525, 1975; Zapolsky, **56**, 1137, 1988).

and force $\mathbf{F}' = q\mathbf{E}'$ can be transformed back to the lab frame, where q appears to be moving with velocity $\mathbf{v}$, as (with no approximation[4])

$$\mathbf{F} = q\mathbf{v} \times \frac{\mu_o I}{2\pi r}\hat{\phi},$$

where $\hat{\phi}$ is the unit vector in the direction given by the *right-hand-rule* (see margin) and $\mu_o = 4\pi \times 10^{-7}$ H/m is *permeability* of free space.

- We find it convenient to define

$$\mathbf{B} \equiv \frac{\mu_o I}{2\pi r}\hat{\phi}$$

to be the "magnetic flux density" of current filament I at a distance r, and attribute the force

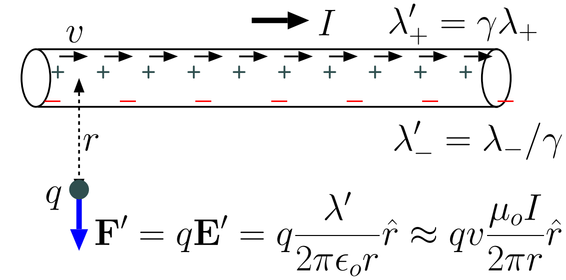
$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

on the moving charge q to the magnetic field $\mathbf{B}$ produced by current I (rather than to the electrostatic field of the wire seen by q in its own reference frame).

While we assumed q to be stationary in the reference frame of the electrons in the above discussion (for the sake of simplicity), the results obtained above are found to be valid for all particle velocities $\mathbf{v}$ measured in the lab frame.

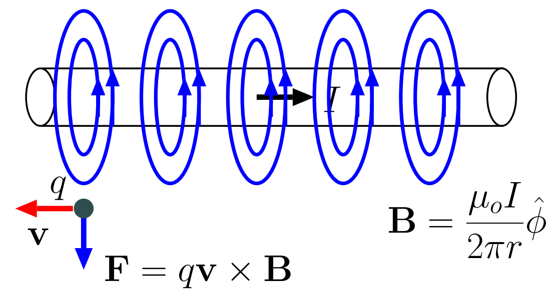
[4]We also get the same result using the approximation $\mathbf{F} = \mathbf{F}'$ that can be justified when $|v| \ll c$, which is typically true by a large margin for electron speeds in current carrying conducting metals — see HW.

(a) In the "electron frame" the wire appears positively charged and repelsa test charge q with force F′=qE′



$$\lambda' \approx \lambda_+\frac{v^2}{c^2} = \frac{Iv}{c^2} = Iv\mu_o\epsilon_o$$

(b) In the lab frame force F~F′ of moving charge q is attributed to magnetic field B produced by current I and velocity v of the charge in F=qvXB combination.



Magnetic field B curls around current I in a right handed direction designated by azimuthal unit vector$\hat{\phi}$

Magnetic field lines close upon themselves unlike electric field lines which start and stop on point charges.

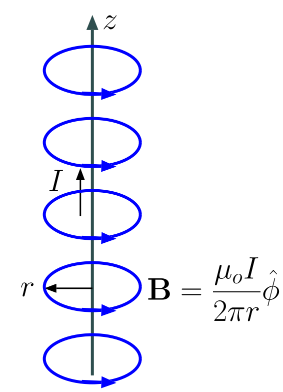
Right hand rule: **point your right thumb in the direction of current flow; your fingers will point in direction $\hat{\phi}$.**

Also, if there are multiple current filaments I_n, each generating its own field $\mathbf{B}_n$, force $\mathbf{F}$ on q can be calculated using a superposition method as with electrostatic fields.

Magnetic field $\mathbf{B}$ of the infinite current filament I obtained above can also be obtained by superposing the magnetic field increments

$$d\mathbf{B} \equiv \frac{\mu_o I d\mathbf{l} \times \hat{r}}{4\pi r^2} \quad \text{(Biot-Savart law)}$$

of directed current increments $Id\mathbf{l}$, where $\mathbf{r} = r\hat{r}$ is a position vector extending from the location of the current increment to the field position where $d\mathbf{B}$ is being specified — this formula, known as Biot-Savart law, is only valid when used in terms of infinitesimal segments $Id\mathbf{l}$ of time-invarying current loops.



- Magnetic field $\mathbf{B}$ of the infinite line current I "derived" above satisfies a circulation relation

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o I_C,$$

with $I_C = I$.

This integral for the circulation of static magnetic field $\mathbf{B}$ is found to be valid (experimentally) for all closed circulation paths C, and is known as **Ampere's law** (for static magnetic fields). In Ampere's law

  - I_C stands for the net sum of all filament currents I_n crossing any surface S bounded by path C,

    - flowing in the direction given by the "right-hand-rule":

when the right thumb is pointed in the direction of $d\mathbf{l}$ along path C, the direction of filament current I_n is specified as the direction of the fingers of your right hand through surface S bounded by contour C.

- Filament currents not crossing S — i.e., current filaments not "linked" to path C — should not be included on the right hand side of Ampere's law.

- Ampere's law can also be expressed as

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S},$$

where

  - we have defined

$$\mathbf{H} \equiv \mu_o^{-1}\mathbf{B}$$

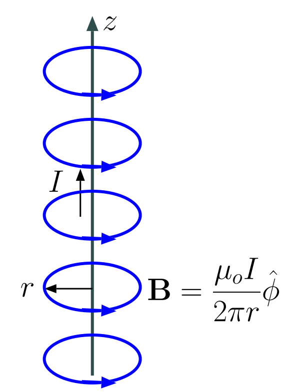
  for the sake of convenience, and

  - $\mathbf{J}$ is the volumetric current density measured in A/m^2 units (e.g., $\sigma\mathbf{E}$ in a conducting region as discussed in last lecture) having a total flux

$$I_C = \int_S \mathbf{J} \cdot d\mathbf{S}$$

  across any surface S bounded by a path C,

    - with $d\mathbf{S}$ pointing across S in the direction compatible with right-hand-rule as in *Stoke's theorem* (recall Lecture 6).



- Stoke's theorem re-stated for a vector field $\mathbf{H}$ as

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{H} \cdot d\mathbf{S}$$

implies that the differential form of Ampere's law should be

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

This differential relation is accompanied by

$$\nabla \cdot \mathbf{B} = 0,$$

satisfied by static magnetic field of the line current *as well as* by any other magnetic field — static as well as non-static, as determined experimentally and described in more detail later on.

- Current density vector field $\mathbf{J}$ invoked above in Ampere's law expressions, measured nominally in units of $\mathrm{A/m^2}$, can also be adjusted to describe the distributions of surface or line currents in 3D space.

  – For example,

$$\mathbf{J}(x, y, z) = \mathbf{J}_s(y, z)\delta(x - x_o)$$

  can be regarded as **volumetric current density** representation of a **surface current density** $\mathbf{J}_s(x, y)$ measured in A/m units flowing on $x = x_o$ surface.

Laws of magnetostatics:

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

They also apply "quasi-statically" over a region of dimension L when a time-varying field source $\mathbf{J}(\mathbf{r}, t)$ has a *time-constant* τ much longer than the propagation time delay L/c of field variations across the region (c is the speed of light).

In magneto-quasistatics (MQS) $\mathbf{B}(\mathbf{r},t) = \mu_o \mathbf{H}(\mathbf{r}, t)$ will be accompanied by a slowly varying electric field $\mathbf{E}(\mathbf{r}, t)$ (derived from Faraday's law discussed in Lecture 14).

– Likewise,

$$\mathbf{J}(x,y,z) = \hat{z}I(z)\delta(x-x_o)\delta(y-y_o)$$

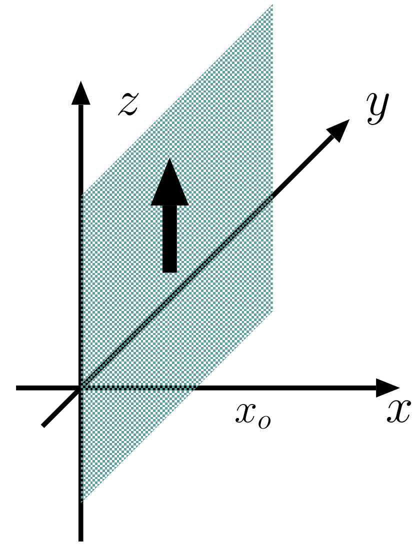
representats a **line current** $I(z)$ measured in A units flowing in z-direction along a filament defined by the intersections of $x = x_o$ and $y = y_o$ surfaces.

– As a most extreme case,

$$\mathbf{J}(x,y,z,t) = Q\mathbf{v}\delta(x-x_o)\delta(y-y_o)\delta(z-z_o)$$

represents the *time-varying* current density of a point charge Q at coordinates $(x,y,z) = (x_o(t), y_o(t), z_o(t))$ moving with velocity $\mathbf{v} = (\dot{x}_o(t), \dot{y}_o(t), \dot{z}_o(t))$.

$\mathbf{J} = \hat{z}\ y\ \mathrm{rect}(y-0.5)\ \delta(x)$



Example 1: Consider a surface current density of

$$\mathbf{J}_s = \hat{z}y\,\mathrm{rect}(y-0.5)\ \mathrm{A/m}$$

flowing on $x = 0$ plane (as shown in the margin). What is the total current I flowing on the same plane measured in A units?

Solution: To go from a surface current density $\mathbf{J}_s$ in A/m to a total current I in A, we need to perform an appropriate integration operation on the surface were $\mathbf{J}_s$ is defined. For the specified $\mathbf{J}_s$ in this problem we find that

$$I = \int_{y=-\infty}^{\infty} \mathbf{J}_s \cdot \hat{z}dy = \int_{y=0}^{1} y\,dy = \frac{y^2}{2}\Big|_0^1 = \frac{1}{2}\ \mathrm{A}.$$