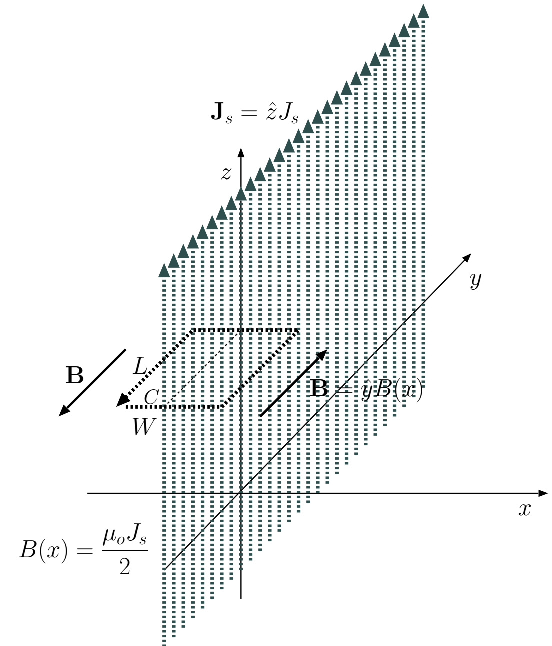
13 Current sheet, solenoid, vector potential and current loops

In the following examples we will calculate the magnetic fields $\mathbf{B} = \mu_o \mathbf{H}$ established by some simple current configurations by using the integral form of static Ampere's law.



As shown in Example 1 magnetic field of a current sheet is independent of distance $|x|$ from the current sheet. Also H changes discontinuously across the current sheet by an amount J_s.

Example 1: Consider a uniform **surface current density** $\mathbf{J}_s = J_s\hat{z}$ A/m flowing on $x = 0$ plane (see figure in the margin) — the current sheet extends infinitely in y and z directions. Determine **B** and **H**.

Solution: Since the current sheet extends infinitely in y and z directions we expect **B** to depend only on coordinate x. Also, the field should be the superposition of the fields of an infinite number of current filaments, which suggests, by right-hand-rule, $\mathbf{B} = \hat{y}B(x)$, where $B(x)$ is an odd function of x. To determine $B(x)$, such that $B(-x) = -B(x)$, we apply Ampere's law by computing the circulation of **B** around the rectangular path C shown in the figure in the margin. We expand

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o I_C$$

as

$$B(x)L + 0 - B(-x)L + 0 = \mu_o J_s L,$$

from which we obtain

$$B(x) = \frac{\mu_o J_s}{2} \;\Rightarrow\; \mathbf{B} = \hat{y}\frac{\mu_o J_s}{2}\mathrm{sgn}(x) \text{ and } \mathbf{H} = \hat{y}\frac{J_s}{2}\mathrm{sgn}(x).$$

Example 2: Consider a slab of thickness W over $-\frac{W}{2} < x < \frac{W}{2}$ which extends infinitely in y and z directions and conducts a uniform current density of $\mathbf{J} = \hat{z}J_o$ A/m^2. Determine $\mathbf{H}$ if the current density is zero outside the slab.

Solution: Given the geometric similarities between this problem and Example 1, we postulate that $\mathbf{B} = \hat{y}B(x)$, where $B(x)$ is an odd function of x, that is $B(-x) = -B(x)$. To determine $B(x)$ we apply Ampere's law by computing the circulation of $\mathbf{B}$ around the rectangular path C shown in the figure in the margin. For $x < \frac{W}{2}$, we expand

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o I_C$$

as

$$B(x)L + 0 - B(-x)L + 0 = \mu_o J_o 2xL \ \Rightarrow\ B(x) = \mu_o J_o x.$$

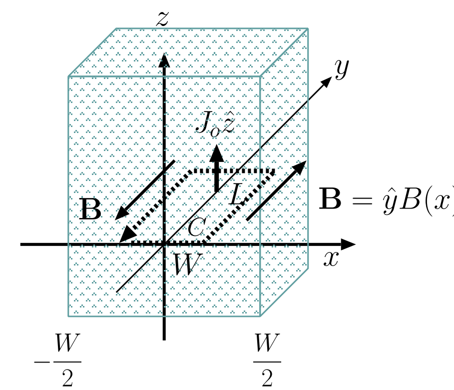
For $x \geq \frac{W}{2}$, the expansion gives

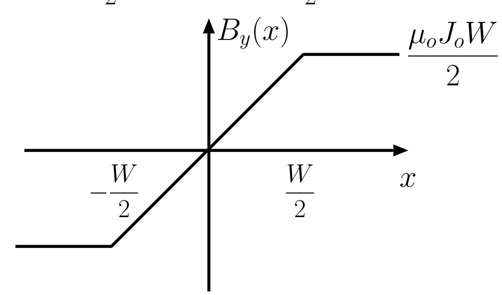
$$B(x)L + 0 - B(-x)L + 0 = \mu_o J_o WL \ \Rightarrow\ B(x) = \mu_o J_o \frac{W}{2}.$$

Hence, we find that

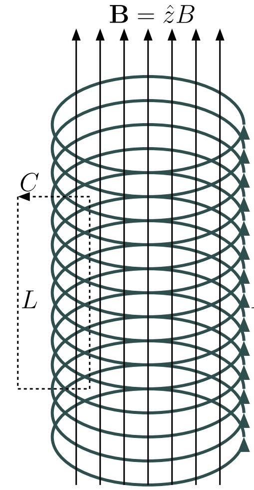
$$\mathbf{H} = \begin{cases} \hat{y}J_o x, & |x| < \frac{W}{2} \\ \hat{y}J_o \frac{W}{2}\,\mathrm{sgn}(x), & \text{otherwise.} \end{cases}$$

Note that the solution plotted in the margin shows no discontinuity at $x = \pm\frac{W}{2}$ or elsewhere.





The figure in the margin depicts a finite section of an infinite solenoid. A solenoid can be constructed in practice by winding a long wire into a multi loop coil as depicted. A solenoid with its loop carrying a current I in $\hat{\phi}$ direction (as shown), produces effectively a surface current density of $\mathbf{J}_s = IN\hat{\phi}$ A/m, where N is the number density (1/m) of current loops in the solenoid. In Example 3 we compute the magnetic field of the infinite solenoid using Ampere's law.



Infinite solenoid with N loops per unit length carrying I amps per loop

$B = \mu_o IN$

Example 3: An infinite **solenoid** having N loops per unit length is stacked in z-direction, each loop carrying a current of I A in counter-clockwise direction when viewed from the top (see margin). Determine $\mathbf{H}$.

Solution: Assuming that $\mathbf{B} = 0$ outside the solenoid, and also $\mathbf{B}$ is independent of z within the solenoid, we find that Ampere's law indicates for the circulation C shown in the margin

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o I_C \quad \Rightarrow \quad LB = \mu_o INL.$$

This leads to

$$B = \mu_o IN \text{ and } \mathbf{H} = \hat{z}IN$$

for the field within the solenoid.

The assumption of zero magnetic flux density $\mathbf{B} = 0$ for the exterior region is justified because:

(a) if the exterior field is non-zero, then it must be independent of x and y (follows from Ampere's law applied to any exterior path C with $I_C = 0$), and

(b) the finite interior flux $\Psi = \mu_o IN\pi a^2$ can only be matched with the flux of the infinitely extended exterior region when the *constant* exterior flux density (because of (a)) is vanishingly small.

- **Static electric fields:** *Curl-free* and are governed by

$$\nabla \times \mathbf{E} = 0, \;\; \nabla \cdot \mathbf{D} = \rho \;\text{ where }\; \mathbf{D} = \epsilon \mathbf{E}$$

  with $\epsilon = \epsilon_r \epsilon_o$.

- **Static magnetic fields:** *Divergence-free* and are governed by

$$\nabla \cdot \mathbf{B} = 0, \;\; \nabla \times \mathbf{H} = \mathbf{J} \;\text{ where }\; \mathbf{B} = \mu \mathbf{H}$$

  with $\mu = \mu_r \mu_o$ — relative *permeabilities* μ_r other than unity (for free space) will be explained later on.

Mathematically, we can generate a **divergence-free** vector field $\mathbf{B}(x, y, z)$ as

$$\mathbf{B} = \nabla \times \mathbf{A}$$

by taking the curl of any vector field $\mathbf{A} = \mathbf{A}(x, y, z)$ (just like we can generate a curl-free $\mathbf{E}$ by taking the gradient of any scalar field $-V(x, y, z)$).

Verification: Notice that

$$\begin{aligned}
\nabla \cdot \nabla \times \mathbf{A} \; &= \; \frac{\partial}{\partial x}(\nabla \times \mathbf{A})_x + \frac{\partial}{\partial y}(\nabla \times \mathbf{A})_y + \frac{\partial}{\partial z}(\nabla \times \mathbf{A})_z = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
&= \; \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = 0.
\end{aligned}$$

- If $\mathbf{B} = \nabla \times \mathbf{A}$ represents a magnetostatic field, then $\mathbf{A}$ is called magnetostatic potential or **vector potential**.

– Vector potential $\mathbf{A}$ can be used in magnetostatics in similar ways to how electrostatic potential V is used in electrostatics.

  ◦ In electrostatics we can assign $V = 0$ *to any point in space* that is convenient in a given problem.

  ◦ In magnetostatics we can assign $\nabla \cdot \mathbf{A}$ *to any scalar* that is convenient in a given problem.

– For example, if we make the assignment[1]

$$\nabla \cdot \mathbf{A} = 0,$$

then we find that

$$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\nabla^2 \mathbf{A}.$$

This is a *nice and convenient* outcome, because, when combined with

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \Rightarrow \quad \nabla \times \mathbf{B} = \mu_o \mathbf{J},$$

it produces

$$\nabla^2 \mathbf{A} = -\mu_o \mathbf{J},$$

which is the magnetostatic version of Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_o}.$$

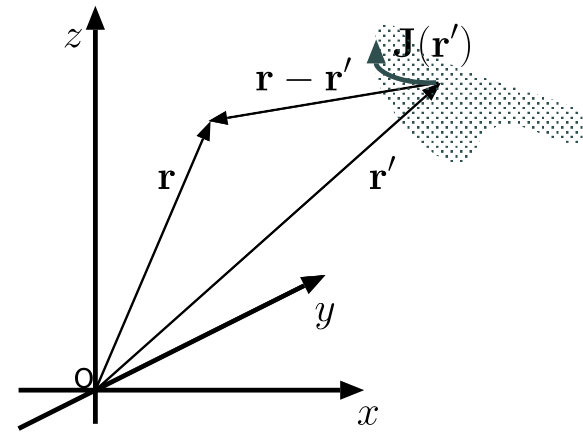
[1]With this assignment — known as *Coulomb gauge* — $\mathbf{A}$ acquires the physical meaning of "potential momentum per unit charge", just as scalar potential V is "potential enegy per unit charge" (see Konopinski, *Am. J. Phys.*, *46*, 499, 1978).

– In analogy with solution

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'$$

of Poisson's equation, it has a solution

$$\mathbf{A}(\mathbf{r}) = \int \frac{\mu_o\mathbf{J}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'.$$

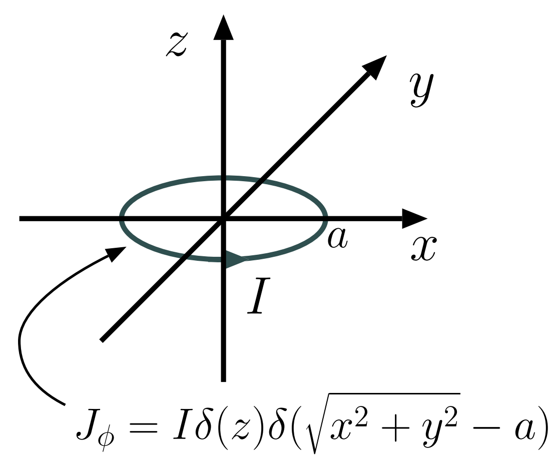


Given *any* static[2] current density $\mathbf{J}(\mathbf{r})$, the above equation can be used to obtain the corresponding vector potential $\mathbf{A}$ that simultaneously satisfies

$$\nabla\cdot\mathbf{A} = 0 \text{ and } \nabla\times\mathbf{A} = \mathbf{B}.$$

Once $\mathbf{A}$ is available, obtaining $\mathbf{B} = \nabla\times\mathbf{A}$ is then just a matter of taking a curl.

- Magnetic flux density $\mathbf{B}$ of a *single* current loop I can be calculated after determining its vector potential as follows:



  – For a loop of radius a on $z = 0$ plane, we can express the corresponding current density as

$$\mathbf{J}(\mathbf{r}') = I\delta(z')\delta(\sqrt{x'^2+y'^2}-a)\frac{(-y',x',0)}{\sqrt{x'^2+y'^2}}$$

  where the ratio on the right is the unit vector $\hat{\phi}'$.

  – Inserting this into the general solution for vector potential, and performing the integration over z', we obtain

[2] Also, in quasi-statics we use $\mathbf{J}(\mathbf{r}',t)$ to obtain $\mathbf{A}(\mathbf{r},t)$ and $\mathbf{B} = \nabla\times\mathbf{A}$ over regions small compared to $\lambda = c/f$, with f the highest frequency in $\mathbf{J}(\mathbf{r}',t)$.

$$
\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \frac{\mu_o I}{4\pi}\int \delta(\sqrt{x'^2+y'^2}-a)\frac{(-y',x',0)}{\sqrt{(x-x')^2+(y-y')^2+z^2}\sqrt{x'^2+y'^2}}dx'dy' \\
&= \frac{\mu_o I}{4\pi}\int \delta(r'-a)\frac{(-y',x',0)}{\sqrt{(x-x')^2+(y-y')^2+z^2}r'}r'dr'd\phi' \\
&= \frac{\mu_o I}{4\pi}\int_{-\pi}^{\pi} \frac{(-a\sin\phi',a\cos\phi',0)}{\sqrt{(x-a\cos\phi')^2+(y-a\sin\phi')^2+z^2}}d\phi' \equiv \hat{x}A_x(\mathbf{r})+\hat{y}A_y(\mathbf{r}).
\end{aligned}
$$

– Given that $A_z = 0$, it can be shown that $\mathbf{B} = \nabla \times \mathbf{A}$ leads to

$$
B_x = -\frac{\partial A_y}{\partial z}, \quad B_y = \frac{\partial A_x}{\partial z}, \quad B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}.
$$

– From the expected azimuthal symmetry of $\mathbf{B}$ about the z-axis, it is sufficient to evaluate these on, say, $y = 0$ plane — after some algebra, and dropping the primes, we find, on $y = 0$ plane,

$$
B_x = \frac{\mu_o a I}{4\pi}\int_{-\pi}^{\pi}\frac{z\cos\phi}{(x^2+a^2+z^2-2ax\cos\phi)^{3/2}}d\phi,
$$

$$
B_y = \frac{\mu_o a I}{4\pi}\int_{-\pi}^{\pi}\frac{z\sin\phi}{(x^2+a^2+z^2-2ax\cos\phi)^{3/2}}d\phi,
$$

and

$$
B_z = \frac{\mu_o a I}{4\pi}\int_{-\pi}^{\pi}\frac{a - x\cos\phi}{(x^2+a^2+z^2-2ax\cos\phi)^{3/2}}d\phi.
$$

– We note that $B_y = 0$ since the B_y integrand above is odd in ϕ and the integration limits are centered about the origin. Hence, the field on $y = 0$ plane is given as

$$
\mathbf{B} = \hat{x}B_x + \hat{z}B_z
$$

with B_x and B_z defined above.

– There are no closed form expressions for the B_x and B_z integrals above for an arbitrary (x, z).

- However, it can be easily seen that if $x = 0$ (i.e., along the z-axis), $B_x = 0$ (as symmetry would dictate) and

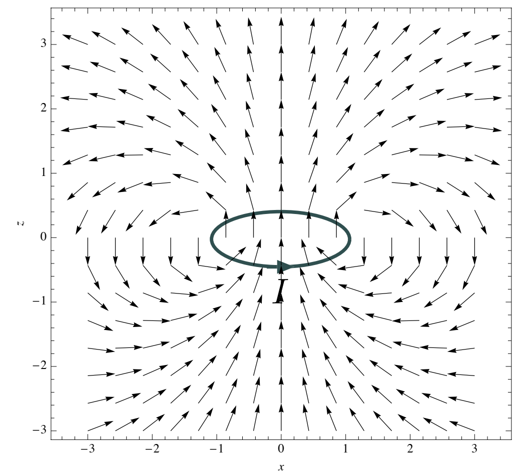
$$B_z = \frac{\mu_o a I}{4\pi} \int_{-\pi}^{\pi} \frac{a}{(a^2+z^2)^{3/2}} d\phi = \frac{\mu_o I a^2}{2(a^2+z^2)^{3/2}}.$$

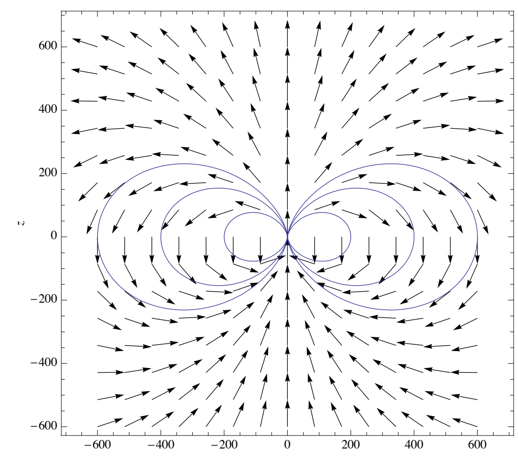
  For $|z| \gg a$,

$$B_z \approx \frac{\mu_o I a^2}{2|z|^3},$$

  which is positive and varies with the inverse third power of distance $|z|$.

– Also, B_x and B_z integrals can be performed numerically. Figure in the margin depicts the pattern of $\hat{B}$ on $y = 0$ plane for a loop of radius $a = 1$ computed using *Mathematica*.



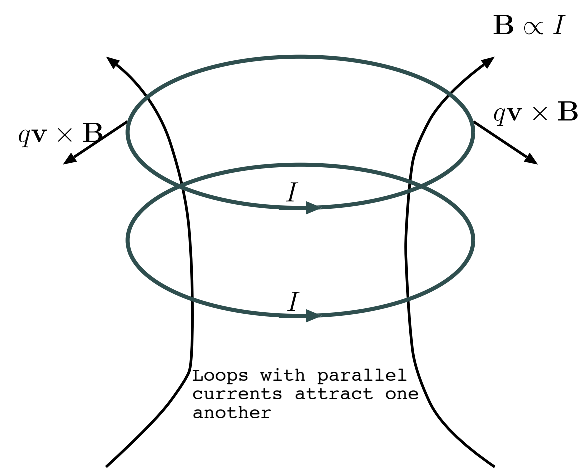


- Note that circulation $\oint_C \mathbf{B} \cdot d\mathbf{l}$ around each closed field line ("linking" the current loop) equals a fixed value of $\mu_o I$ — this dictates that the average field strength $|\mathbf{B}|$ of a current loop is stronger on shorter field lines closer to the current loop than on longer field lines linking the loop further out. As a result $|\mathbf{B}|$ can be shown to vary as r^{-3} for large r.

- It can be shown that the equations for magnetic *field lines* of a current loop on, say, $y = 0$ plane, can be expressed as

$$r = L \sin^2 \theta$$

in terms of **radial distance** r from the origin and **zenith angle** θ measured from the z axis. Clearly, parameter L in this formula is the radial distance of the field line on $\theta = 90^o$ plane, and the field line formula is accurate only for $r \gg a$. The Earth's magnetic field had such a **magnetic dipole** topology as shown.



- Lorentz force due to the magnetic fields of a pair of current loops — also known as *magnetic dipoles* — turns out to be "attractive" when the current directions agree (see margin). Bar magnets carrying "equivalent" current loops of atomic origins interact with one another in exactly the same way — i.e., as governed by the second term of Lorentz force.