## 18 Wave equation and plane TEM waves in sourcefree media

With this lecture we start our study of the full set of Maxwell's equations shown in the margin by first restricting our attention to *homogeneous* and *non-conducting* media with constant  $\epsilon$  and  $\mu$  and zero  $\sigma$ .

- Our first objective is to show that non-trivial (i.e., non-zero) timevarying field solutions of these equations can be obtained even in the absence of  $\rho$  and **J**.
  - We already know *static*  $\rho$  and **J** to be the **source** of *static* electric and magnetic fields.
  - We will come to understand that time varying  $\rho$  and **J**, which necessarily obey the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

constitute the **source** of *time-varying electromagnetic* fields.

Despite these intimate connections between the sources  $\rho$  and  ${\bf J}$  and the fields

$$\mathbf{D} = \epsilon \mathbf{E}$$
 and  $\mathbf{B} = \mu \mathbf{H}$ ,

*non-trivial field solutions can exist* in source-free media as we will see shortly.

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 $\nabla \cdot \mathbf{D} = \rho$  $\nabla \cdot \mathbf{B} = 0$  $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$  $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$ 

- Such field solutions in fact represent electromagnetic waves, a familiar example of which is **light**.
- Another example is **radiowaves** that we use when we communicate using wireless devices such as radios, cell-phones, WiFi, etc.
- Different types of electromagnetic waves are distinguished by their oscillation frequencies, and include
  - radiowaves,
  - microwaves,
  - infrared,
  - light,
  - ultraviolet,
  - X-rays, and gamma rays,

going across the **electromagnetic spectrum** from low to high frequencies.

We are well aware that these types of electromagnetic waves can travel across empty regions of space — e.g., from sun to Earth — transporting energy and heat as well as momentum.

- Next, we will discover their general properties by examining Maxwell's equations under the restriction  $\rho = \mathbf{J} = 0$ .

- In source-free and homogeneous regions where  $\rho = \mathbf{J} = 0$  and  $\epsilon$  and  $\mu$  are constant, we can simplify Maxwell's equations as shown in the margin.
  - If there are non-trivial solutions of these equations, namely  $\mathbf{E}(\mathbf{r}, t) \neq 0$  and  $\mathbf{H}(\mathbf{r}, t) \neq 0$ , they evidently need to be divergence-free.
  - They also have to be "curly" according to the last two equations: Faraday's and Ampere's laws.
- Next we will make use of vector identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

which should be familiar from an earlier homework problem.

- Since the electric field  $\mathbf{E}$  is divergence-free in the absence of sources, this identity simplifies as

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}$$

where in the right side  $\nabla^2 \mathbf{E}$  is the Laplacian of  $\mathbf{E}$ .

- Using this result we can express the curl of Faraday's law as

$$\nabla \times [\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}] \;\; \Rightarrow \;\; -\nabla^2 \mathbf{E} = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H},$$

which combines with the Ampere's law to produce

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$$\nabla^2 \mathbf{E} = \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2},$$

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{H} = 0$$
$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$
$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}.$$

which can be written explicitly as

3D vector wave equation

Recall that our objective is to see whether a non-trivial time-varying solution of Maxwell's equations can exist in source-free media.

 $\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} = \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}.$ 

Our objective at this stage is not finding a general solution; it is instead identifying a simple example of a non-trivial time-varying  $\mathbf{E}(\mathbf{r}, t)$ , if we can.

For example, can a field solution

$$\mathbf{E}(\mathbf{r},t) = \hat{x}E_x(z,t)$$

that only depends on z and t and "polarized" in x-direction exist? If it can exist, what would be the properties of this x-polarized solution?

• To find out, we note that with  $\mathbf{E} = \hat{x} E_x(z, t)$ , the above "wave equation" is reduced to  $\partial^2 E = \partial^2 E$  **1D scalar** 

$$\frac{\partial^2 E_x}{\partial z^2} = \mu \epsilon \frac{\partial^2 E_x}{\partial t^2}, \qquad \qquad \text{wave} \\ \text{equation}$$

an equation that is known as a **1D scalar wave equation**, as opposed to the **3D** *vector* wave equation above.

- Now, by substitution, we can easily show that

$$E_x = \cos(\omega(t - \sqrt{\mu\epsilon}z)),$$

satisfies the 1D wave equation and represents an x-polarized timeperiodic field solution with an oscillation frequency  $\omega$ .

- 1D wave equation can also be satisfied by

$$E_x = \cos(\omega(t + \sqrt{\mu\epsilon}z)).$$

Let us jointly refer to these solutions as

$$E_x = \cos(\omega(t \mp \frac{z}{v})),$$

where

$$v \equiv \frac{1}{\sqrt{\mu\epsilon}}$$

has the dimensions of m/s (i.e., velocity) and the algebraic signs  $\mp$  distinguish between the "travel directions" of these *possible* "wave solutions" as elaborated later on.

• Let us next find out the magnetic field intensity  $\mathbf{H}$  that accompanies the *x*-polarized electric field wave solution

$$\mathbf{E} = \hat{x}\cos(\omega(t\mp\frac{z}{v})).$$

– Since the curl of  ${\bf E}$  is

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \hat{y} \frac{\partial E_x}{\partial z} = \pm \hat{y} \sin(\omega(t \mp \frac{z}{v})) \frac{\omega}{v},$$

Faraday's law

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

requires that  $\mathbf{H}$  should satisfy

$$-\mu \frac{\partial \mathbf{H}}{\partial t} = \pm \hat{y} \sin(\omega (t \mp \frac{z}{v})) \frac{\omega}{v}.$$

Finding the *time-dependent* anti-derivative (and remembering  $v = 1/\sqrt{\mu\epsilon}$ ), we obtain

$$\mathbf{H} = \pm \hat{y} \sqrt{\frac{\epsilon}{\mu}} \cos(\omega(t \mp \frac{z}{v})).$$

• The results above, namely our *x*-polarized non-trivial field solutions of Maxwell's equations in source-free homogeneous space, can be represented more compactly as

$$\mathbf{E} = \hat{x}f(t \mp \frac{z}{v}) \text{ and } \mathbf{H} = \pm \hat{y}\frac{f(t \mp \frac{z}{v})}{\eta},$$

where

$$f(t) \equiv \cos(\omega t) = \operatorname{Re}\{e^{j\omega t}\} = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

is the field waveform,

$$\eta \equiv \sqrt{\frac{\mu}{\epsilon}}$$

is known as **intrinsic impedance** (and measured in units of ohms).





• Since Maxwell's equations with constant  $\mu$  and  $\epsilon$  are linear and timeinvariant (LTI), the field solutions above can be further generalized by using their weighted and time-shifted superpositions such as

$$f(t) = \sum_{n} A_n \cos(\omega_n t + \theta_n)$$

and

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

having frequency dependent weighting factors  $A_n$  and  $F(\omega)$ . And since according to Fourier analysis all practical signals f(t) can be synthesized in these forms, it follows that the field solutions above are valid with *arbitrary* waveforms f(t).

Solutions

$$\mathbf{E},\,\mathbf{H}\propto f(t\mp\frac{z}{v})$$

d'Alembert

wave solutions

of the 1D scalar wave equation with arbitrary f(t) are known as **d'Alembert** wave solutions.

• d'Alembert solution

$$\mathbf{E}, \, \mathbf{H} \propto f(t - \frac{z}{v})$$

describes electromagnetic waves traveling in +z direction, whereas solution

$$\mathbf{E}, \, \mathbf{H} \propto f(t + \frac{z}{v})$$

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describes electromagnetic waves traveling in -z direction (see margin). In each case the travel speed is

$$v = \frac{1}{\sqrt{\mu\epsilon}} \quad \overrightarrow{\text{free space}} \quad \frac{1}{\sqrt{\mu_o\epsilon_o}} \equiv c \approx 3 \times 10^8 \, \text{m/s}$$

- **H** solution can be obtained from **E** by dividing it with  $\eta$  and rotating it by 90° so that vector **E** × **H** points in direction the waves travel.
- E can be obtained from H by multiplying it with η and rotating it by 90° so that vector E × H called Poynting vector once again points in direction the waves travel.

In each case the intrinsic impedance is

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \quad \overrightarrow{\text{free space}} \quad \sqrt{\frac{\mu_o}{\epsilon_o}} \equiv \eta_o \approx 120\pi \text{ ohms.}$$

Transformation rules above also hold for y-polarized wave solutions

$$\mathbf{E} = \hat{y}f(t \mp \frac{z}{v})$$
 and  $\mathbf{H} = \mp \hat{x}\frac{f(t \mp \frac{z}{v})}{\eta}$ .

Question: What about *z*-polarized waves

$$\mathbf{E} = \hat{z}f(t \mp \frac{z}{v}),$$

can they exist?

**Answer:** No, z-polarized waves  $\hat{z}f(t \mp \frac{z}{v})$  traveling in  $\pm z$  direction cannot exist because they would violate the divergence-free condition  $\nabla \cdot \mathbf{E} = 0$ .

ct=300 km in 1 millisec

Position plots at t=0 and t>0





Traveling wave in +z direction with speed v=c:

