19 d'Alembert wave solutions, radiation from current sheets

- d'Alembert wave solutions of Maxwell's equations for homogeneous and source-free regions obtained in the last lecture having the forms

$$\mathbf{E},\ \mathbf{H} \propto f(t \mp \frac{z}{v})$$

are classified as **uniform plane-TEM waves**.

  - TEM stands for **T**ransverse **E**lectro**M**agnetic, and the reason for this designation is:

viable solutions satisfying $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{H} = 0$ conditions have their $\mathbf{E}$ and $\mathbf{H}$ vectors **transverse** to the *direction of propagation* which always coincides with the direction of vector $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$ known as **Poynting vector** — more on this later on.

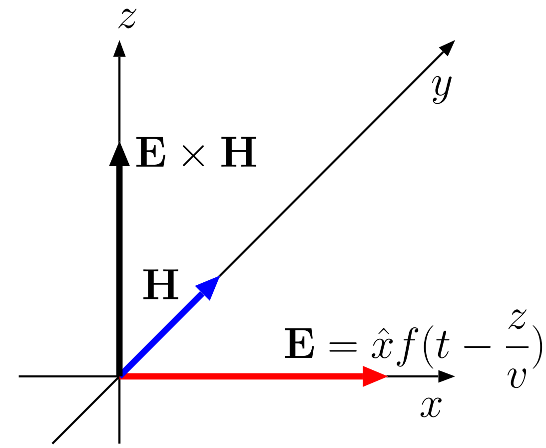
  - d'Alembert wave solutions such as

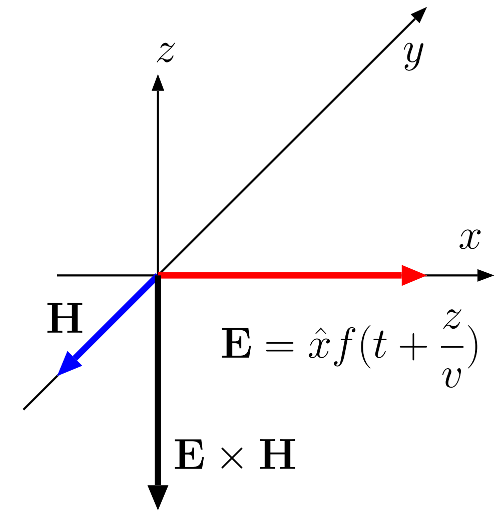
$$\mathbf{E} = \hat{x} f(t - \frac{z}{v}) \quad \text{and} \quad \mathbf{H} = \hat{y} \frac{f(t - \frac{z}{v})}{\eta}$$

are also designated as *uniform plane* waves because:

these wave-fields are constant (have the same vector value) at **planes of constant phase**, e.g., on planes defined by

$$t - \frac{z}{v} = \text{const.},$$





Poynting vector $\mathbf{E} \times \mathbf{H}$

which are planes transverse to the propagation direction (direction of vector $\mathbf{E} \times \mathbf{H}$).

Not all waves solutions of Maxwell's equations are uniform plane — for instance non-uniform TEM waves with spherical surfaces of constant phase are ubiquitous, but they will be examined later on (in ECE 450, mainly).

After the next set of examples we will examine how uniform plane waves can be radiated by infinite planes of surface currents. By contrast, spherical waves are produced by compact antennas having finite dimensions.

Example 1: Let

$$\mathbf{E} = \hat{x}\triangle(\frac{t - y/c}{\tau})$$

be a wave solution in free space where $\triangle(\frac{t}{\tau})$ is a triangular waveform of duration τ peaking at $t = 0$ (defined in ECE 210). We will next provide two different solutions demonstrating how the wave field $\mathbf{B}$ accompanying $\mathbf{E}$ can be found.

Solution 1: We recognize the given wave field $\mathbf{E}$ as a TEM uniform plane wave traveling in y-direction given the $t - y/c$ dependence of phase. Consequently, we obtain $\mathbf{H}$ by dividing $\mathbf{E}$ with $\eta = \eta_o$ and rotating it by $90°$ from $\hat{x}$-direction to co-align it with $\mathbf{E} \times \mathbf{H}$ vector. As a result,

$$\mathbf{H} = -\hat{z}\frac{\triangle(\frac{t-y/c}{\tau})}{\eta_o} = -\hat{z}\frac{\triangle(\frac{t-y/c}{\tau})}{\sqrt{\mu_o/\epsilon_o}}.$$

Hence,

$$\mathbf{B} = \mu_o\mathbf{H} = -\hat{z}\sqrt{\mu_o\epsilon_o}\triangle(\frac{t - y/c}{\tau}) = -\hat{z}\frac{\triangle(\frac{t-y/c}{\tau})}{c}.$$

Solution 2: According to Faraday's law,

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E} = -\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & 0 & 0\end{vmatrix} = \hat{z}\frac{\partial E_x}{\partial y}\\ &= \hat{z}\triangle'(\frac{t-y/c}{\tau})\frac{\partial}{\partial y}(\frac{t-y/c}{\tau}) = \hat{z}\frac{-1}{c\tau}\triangle'(\frac{t-y/c}{\tau})\end{aligned}$$

with the help of chain rule of differentiation, where $\triangle$'$(u)\equiv\frac{d}{du}\triangle(u)$. Finding the time-dependent anti derivative, we directly obtain (as before)

$$\mathbf{B} = -\hat{z}\frac{\triangle(\frac{t-y/c}{\tau})}{c}.$$

Example 2: Consider the Lorentz force

$$\mathbf{F} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

on a test charge q in the lab where $\mathbf{E}$ and $\mathbf{B}$ are the plane wave fields considered in Example 1. Show that electrical force term $q\mathbf{E}$ will dominate the magnetic force term $q\mathbf{v}\times\mathbf{B}$ unless the particle speed $v=|\mathbf{v}|$ is close to the speed of light c (i.e., test charge is relativistic).

Solution: Since

$$\mathbf{E} = \hat{x}\triangle(\frac{t-y/c}{\tau}) \quad \text{and} \quad \mathbf{B} = -\hat{z}\frac{\triangle(\frac{t-y/c}{\tau})}{c},$$

it follows that Lorentz force

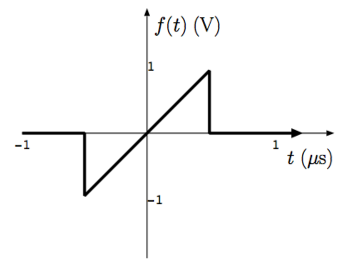
$$\mathbf{F} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = q\triangle(\frac{t-y/c}{\tau})(\hat{x}-\mathbf{v}\times\frac{\hat{z}}{c}).$$

Clearly, the first term of $\mathbf{F}$ proportional to $\hat{x}$ is dominant, unless $v = |\mathbf{v}|$ is close to c.

Example 3: Consider an $\hat{x}$-polarized plane TEM wave field in free space propagating in $+z$ direction such that

$$\mathbf{E}(z,t) = \hat{x} f(t - \frac{z}{c}), \text{ with } f(t) = At\,\mathrm{rect}(\frac{t}{\tau}),$$

where $c = 3 \times 10^8\,\mathrm{m/s} = 300\,\mathrm{m}/\mu\mathrm{s}$ is the speed of light in free space, $\tau = 1\,\mu\mathrm{s}$, and $A = 2\,\frac{\mathrm{V/m}}{\mu\mathrm{s}}$. A plot of $f(t)$ vs t (labelled in μs units is shown in the margin. Determine the corresponding $\mathbf{H}(z,t)$ and make the following plots:

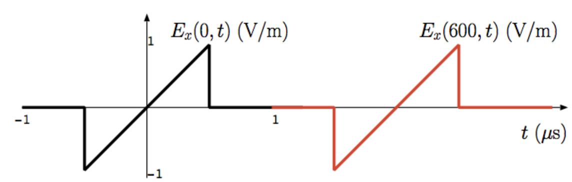


- (a) t-plots at fixed z's: $E_x(0,t)$ and $E_x(z = 600\,\mathrm{m}, t)$,
- (b) z-plots at fixed t's: $E_x(z,0)$ and $E_x(z, 2\,\mu\mathrm{s})$,

Solution:

(a) t-plots at fixed z's: Since $z/c = 2\,\mu\mathrm{s}$ for $z = 600$ m, it follows that

$$E_x(600\,\mathrm{m}, t) = 2\,(t - 2\mu\mathrm{s})\,\mathrm{rect}(\frac{t - 2\mu\mathrm{s}}{1\mu\mathrm{s}})\,\frac{\mathrm{V}}{\mathrm{m}}$$



is a shifted version of

$$E_x(0,t) = 2\,t\,\mathrm{rect}(\frac{t}{1\mu\mathrm{s}})\,\frac{\mathrm{V}}{\mathrm{m}}$$

already plotted above. A graph showing both waveforms (black for $z = 0$ and red for $z = 600$ m) is in the margin.

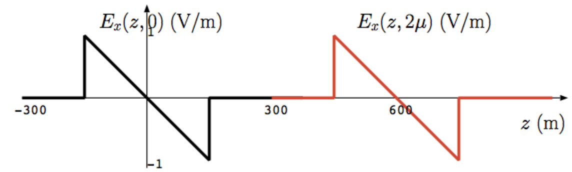
(b) z-plots at fixed t's: In this case we wish to deptict

$$E_x(z,0) = 2\,(0 - \frac{z}{c})\,\mathrm{rect}(\frac{0 - z/c}{1\mu})\,\frac{\mathrm{V}}{\mathrm{m}}$$

and

$$E_x(z, 2\,\mu\mathrm{s}) = 2\,(2\mu - \frac{z}{c})\,\mathrm{rect}(\frac{2\mu - z/c}{1\mu})\,\frac{\mathrm{V}}{\mathrm{m}}.$$

The minus sign in front of z in the first term on the right indicates that the slopes of the curves to be plotted are negative. Hence, we end up with the descending ramp waveforms (black for $t = 0$ and red for $t = 2\mu$s) shown in the margin.



- Plane electromagnetic waves discussed above propagate in free-space in regions of zero ρ and $\mathbf{J}$ (per our derivation).
  - But what generates such waves?
- The answer must be, far away ρ and $\mathbf{J}$ variations (linked by continuity equation) that we have not considered in our equations so far.
- We will next describe how plane TEM waves can be produced — radiated — by time-varying infinite current sheets by starting from familiar static and quasi-static solutions:

- Consider first a static and constant surface current density

$$\mathbf{J}_s = \hat{x} J_x \,\mathrm{A/m}$$

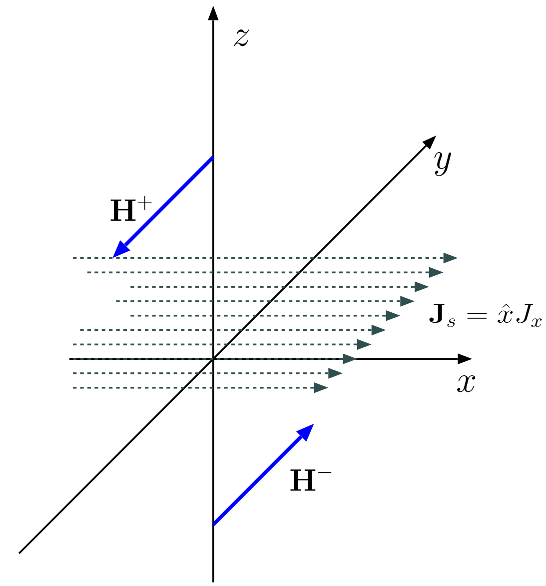
flowing on $z = 0$ surface as shown graphically in the margin. This infinite surface current will produce a static magnetic field

$$\mathbf{H}(z) = \mp \hat{y} \frac{J_x}{2} \,\mathrm{A/m} \text{ for } z \gtrless 0$$

also shown in the margin as we learned in Lecture 13.

  – Note that the fields point in opposing directions above and below the surface current in compliance with the right hand rule and obey the boundary condition equation for tangential $\mathbf{H}$.

  – Also, $\mathbf{H}$ is not accompanied by an electric field $\mathbf{E}$ since static currents produce only static magnetic fields.



- What if the surface current J_x varies with time, i.e., $J_x = J_x(t)$. In that case we have quasi-statically

$$\mathbf{H}(z,t) \approx \mp \hat{y} \frac{J_x(t)}{2} \,\mathrm{A/m} \text{ for } z \gtrless 0,$$

but only as an approximation for positions very close to $z = 0$ surface where propagation time-delay $\frac{z}{v}$ of d'Alembert solutions can be neglected[1].

[1]This solution surely cannot be an exact solution since if it were, it would imply instantaneous changes in $\mathbf{H}$ in response J_x at arbirarily large distances, implying an infinite speed of propagation — we know that is not true!

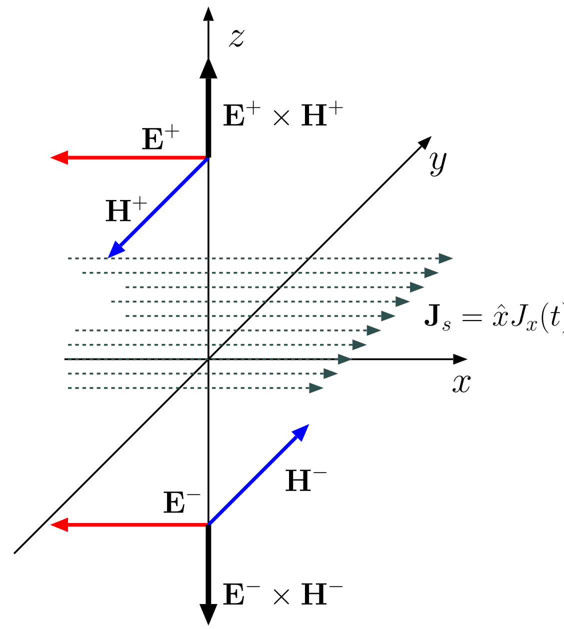
- But the *exact* field solution of Maxwell's equtions valid for all z is equally easy to obtain: just replace $J_x(t)$ above with $J_x(t \mp \frac{z}{v})$ and replace $\approx$ with $=$ so that

$$\mathbf{H}(z,t) = \mp\hat{y}\frac{J_x(t \mp \frac{z}{v})}{2}\text{ A/m for } z \gtrless 0$$

complies with plane TEM d'Alembert solutions[2] of Maxwell's equations in homogeneous and sourece free regions $z \gtrless 0$.

- As always, there is an accompanying $\mathbf{E}(z,t)$ that is obtained by multiplying $\mathbf{H}(z,t)$ with η and replacing its unit vector so that vector $\mathbf{E}\times\mathbf{H}$ points in the direction of propagation, away from the $z=0$ in this case — hence, as illustrated in the margin,

$$\mathbf{E}(z,t) = -\hat{x}\frac{\eta}{2}J_x(t \mp \frac{z}{v})\text{ V/m for } z \gtrless 0.$$



Since Maxwell's eqn's + boundary conditions have *unique* solutions in given settings, we are assured that any solution that complies with both (as in this case) is *the* solution for the given setting (surface current on $z=0$, in this case) — it was surprisingly easy to solve this radiation problem by starting from simple static and quasi-static solutions.

[2]We use $J_x(t \mp \frac{z}{v})$ rather than $J_x(t \pm \frac{z}{v})$ for $z \gtrless 0$ because we assume that $J_x(t)$ on $z=0$ surface is the only field source — in that case *causality* principle dictates that we use only the solutions propagating *away* from the source (just like when a pebble drops in a pond, ripples propagate *away*).

Conclusion: Evidently, a time varying surface current

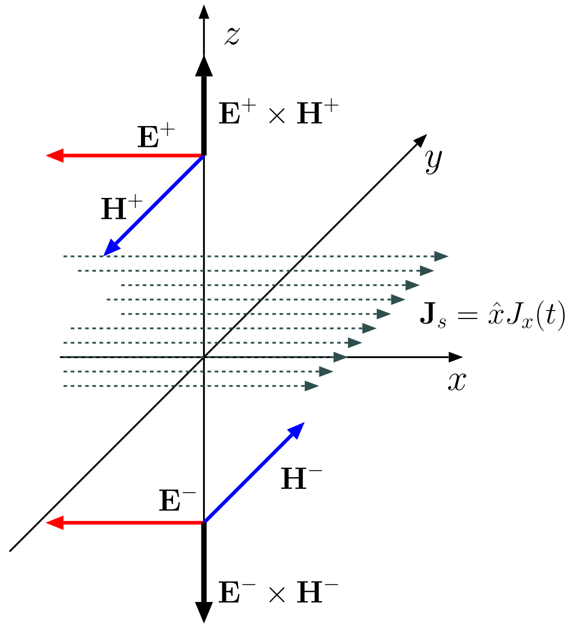
$$\mathbf{J}_s = \hat{x} f(t) \text{ on } z = 0 \text{ plane}$$

produces plane electromagnetic waves

$$\mathbf{E}^{\pm} = -\hat{x}\frac{\eta f(t \mp \frac{z}{v})}{2} \text{ and } \mathbf{H}^{\pm} = \mp\hat{y}\frac{f(t \mp \frac{z}{v})}{2} \text{ in regions } z \gtrless 0$$

propagating away from the $z = 0$ plane.

Note that:



1. E_x and H_y waveforms are proportional to delayed versions of surface current $J_x(t)$ at each location z above and below the current sheet, with the reference directions of $\mathbf{E}$ and $\mathbf{J_s}$ *opposing* one another.

2. fields $\mathbf{E}^{\pm}$ are continuous on $z = 0$ surface in compliance with tangential boundary condition equations.

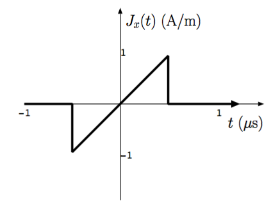
3. fields $\mathbf{H}^{\pm}$ exhibit a discontinuity on $z = 0$ surface that matches the current density of the same surface, once again in compliance with tangential boundary condition equations.

Opposing $\mathbf{E}$ and $\mathbf{J}_s$ vectors on $z = 0$ plane indicate that the surface is acting as a source of radiated energy (the energy that feeds the waves radiated away from the surface) — this interpretation will be discussed in more detail in the next lecture.

Example 4: A current sheet on $z = 0$ surface is described by

$$\mathbf{J}_s(t) = \hat{x} f(t), \quad \text{with} \quad f(t) = At\,\mathrm{rect}(\frac{t}{\tau}),$$

where $\tau = 1\,\mu$s and $A = 2\,\frac{\mathrm{A/m}}{\mu\mathrm{s}}$. A plot of the current waveform $f(t)$ is plotted in the margin. Assuming that the current sheet is embedded in free space, construct the following plots:



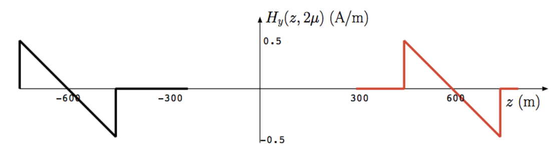
- (a) Radiated $H_y(z, t = 2\mu\mathrm{s})$ vs z,
- (b) Radiated$E_x(z, t = 2\mu\mathrm{s})$ vs z.

Solution:

(a) From the theory developed above, we have using delayed copies of half the surface current density,

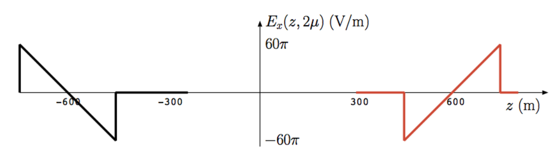
$$H_y(z, 2\mu\mathrm{s}) = \mp(2\mu \mp \frac{z}{c})\,\mathrm{rect}(\frac{2\mu \mp \frac{z}{c}}{1\mu})\,\frac{\mathrm{A}}{\mathrm{m}} \text{ for } z \gtrless 0,$$

as plotted in the margin. Notice that the propagated field waveforms — $c \times 2\mu$s=600 m has been covered in 2 μs — are re-scaled and shifted replicas of the source function $f(t)$.



(b) We have, multiplying H_y with $\eta_o = 120\pi\,\Omega$, and adjusting the signs so that $\mathbf{E}$ and $\mathbf{J_s}$ are pointing in opposite directions,

$$E_x(z, 2\mu\mathrm{s}) = -120\pi(2\mu \mp \frac{z}{c})\,\mathrm{rect}(\frac{2\mu \mp \frac{z}{c}}{1\mu})\,\frac{\mathrm{V}}{\mathrm{m}} \text{ for } z \gtrless 0.$$



Plots are shown in the margin.