20 Poynting theorem and monochromatic waves

- The magnitude of **Poynting vector**

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

represents the amount of power transported — often called energy flux — by electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ over a unit area transverse to the $\mathbf{E} \times \mathbf{H}$ direction.

This interpretation of the Poynting vector is obtained from a conservation law extracted from Maxwell's equations (see margin) as follows:

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \end{aligned}$$

1. Dot multiply Faraday's law by $\mathbf{H}$, dot multiply Ampere's law by $\mathbf{E}$,

$$\begin{aligned} (\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}) \cdot \mathbf{H} \\ (\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \cdot \mathbf{E} \end{aligned}$$

and take their difference:

$$\underbrace{\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}}_{\nabla \cdot (\mathbf{E} \times \mathbf{H})} = \underbrace{-\frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{H}}_{-\frac{\partial}{\partial t}(\frac{1}{2}\epsilon \mathbf{E} \cdot \mathbf{E} + \frac{1}{2}\mu \mathbf{H} \cdot \mathbf{H})} - \mathbf{J} \cdot \mathbf{E}.$$

2. After re-arrangements shown above, the result can be written as

$$\frac{\partial}{\partial t}(\frac{1}{2}\epsilon\mathbf{E}\cdot\mathbf{E}+\frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H})+\nabla\cdot(\mathbf{E}\times\mathbf{H})+\mathbf{J}\cdot\mathbf{E}=0.$$

- **Poynting theorem** derived above is a *conservation law* just like the continuity equation $\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}=0$: **Poynting theorem**

  - The first term on the left,

  $$\frac{\partial}{\partial t}(\frac{1}{2}\epsilon\mathbf{E}\cdot\mathbf{E}+\frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H}),$$

  is time rate of change of total electric and magnetic **energy** density.

  Hence, **Poynting theorem is the conservation law for electromagnetic energy**, just like continuity equation is the conservation law for electric charge.

  - The second term

  $$\nabla\cdot(\mathbf{E}\times\mathbf{H})$$

  accounts for energy transport in Poynting theorem, just like $\nabla\cdot\mathbf{J}$ accounts for charge transport in the continuity equation. Therefore

  $$\mathbf{S}\equiv\mathbf{E}\times\mathbf{H}$$

is **energy flux per unit area** measured in

$$\frac{\mathrm{V}}{\mathrm{m}}\frac{\mathrm{A}}{\mathrm{m}} = \frac{\mathrm{W}}{\mathrm{m}^2} = \frac{\mathrm{J/s}}{\mathrm{m}^2}$$

units, just like $\mathbf{J}$ is charge flux per unit area in $\frac{\mathrm{C/s}}{\mathrm{m}^2} = \frac{\mathrm{A}}{\mathrm{m}^2}$ units.

– Finally, the last term in Poynting theorem (repeated in the margin),

$$\mathbf{J}\cdot\mathbf{E}$$

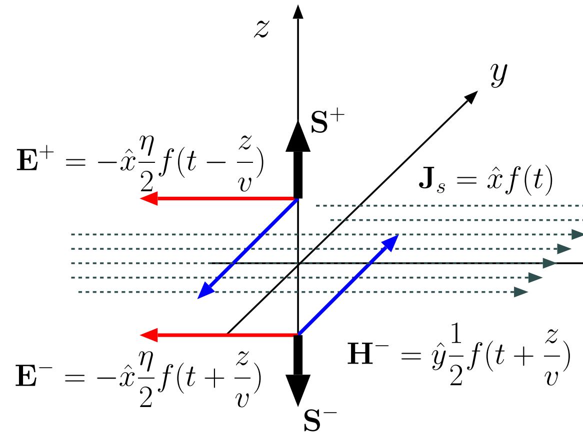
is called **Joule heating**, and it represents power absorbed per unit volume (which can only be non-zero in the presence of $\mathbf{J}$).

If $\mathbf{J}\cdot\mathbf{E}$ is negative in any region, then $\mathbf{J}$ in that region is acting as a source of electromagnetic energy, just like any circuit component with negative vi is acting as an energy source in the electrical circuit.

Note that $\mathbf{J}\cdot\mathbf{E}$ had a negative value on the current sheet radiator examined in last lecture. We return to the current sheet radiator in the next example.

Poynting thm:

$\frac{\partial}{\partial t}(\frac{1}{2}\epsilon\mathbf{E}\cdot\mathbf{E}+\frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H})+\nabla\cdot(\mathbf{E}\times\mathbf{H})+\mathbf{J}\cdot\mathbf{E}=0$



Example 1: On $z = 0$ plane we have a time-harmonic surface current specified as

$$\mathbf{J}_s = \hat{x}f(t) = \hat{x}2\,\cos(\omega t)\,\frac{\mathrm{A}}{\mathrm{m}}$$

where ω is some frequency of oscillation.

(a) Determine the radiated TEM wave fields $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ in the regions $z \gtrless 0$,

(b) The associated Poynting vectors $\mathbf{E} \times \mathbf{H}$, and

(c) $\mathbf{J}_s \cdot \mathbf{E}$ on the current sheet.

Solution:

(a) With reference to the solution of the current sheet radiator depicted in the margin (from last lecture), we that an x-polarized surface current $f(t)$ produces the wave fields

$$E_x = -\frac{\eta}{2}f(t \mp \frac{z}{v}) \quad \text{and} \quad H_y = \mp\frac{1}{2}f(t \mp \frac{z}{v})$$

in the surrounding regions propagating away from the current sheet on both sides. Given that $f(t) = 2\cos(\omega t)$, this implies that

$$E_x = -\eta\cos(\omega t \mp \beta z) \quad \text{and} \quad H_y = \mp\cos(\omega t \mp \beta z)$$

where

$$\beta = \frac{\omega}{c} \quad \text{and} \quad \eta = \eta_o \approx 120\pi\,\Omega$$

since the current sheet is surrounded by vacuum. Hence in vector form we have

$$\mathbf{E}(z,t) = -\eta\cos(\omega t \mp \beta z)\hat{x}\,\frac{\mathrm{V}}{\mathrm{m}} \quad \text{and} \quad \mathbf{H}(z,t) = \mp\cos(\omega t \mp \beta z)\hat{y}\,\frac{\mathrm{A}}{\mathrm{m}},$$

where the upper signs are for $z > 0$, and lower signs for $z < 0$.

(b) The associated Poynting vectors are

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \pm\eta \cos^2(\omega t \mp \beta z)\hat{z}\, \frac{\mathrm{W}}{\mathrm{m}^2}.$$

Note that the time-average value of vector $\mathbf{S}$ points in the direction of wave propagation on both sides of the current sheet.

(c) Since on $z = 0$ surface of the current sheet the electric field vector is

$$\mathbf{E}(0,t) = -\eta \cos(\omega t)\hat{x}\, \frac{\mathrm{V}}{\mathrm{m}},$$

it follows that $\mathbf{J}_s \cdot \mathbf{E}$ on the same surface is

$$\mathbf{J}_s(t) \cdot \mathbf{E}(0,t) = (\hat{x}2\, \cos(\omega t)\, \frac{\mathrm{A}}{\mathrm{m}}) \cdot (-\eta \cos(\omega t)\hat{x}\, \frac{\mathrm{V}}{\mathrm{m}}) = -2\eta \cos^2(\omega t)\, \frac{\mathrm{W}}{\mathrm{m}^2}.$$

- In the above example, a time-harmonic source current oscillating at some frequency ω produced "monochromatic waves" of radiated fields propagating away from the current sheet on both sides.

  - The calculations showed time-varying Poynting vectors $\mathbf{E} \times \mathbf{H}$.

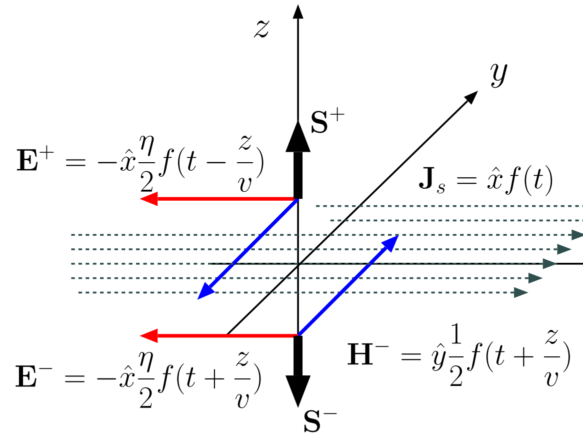
    The time-averaged values of these time-varying vectors can be easily determined by making use of the trig identity

$$\cos^2(\omega t + \phi) = \frac{1}{2}[1 + \cos(2\omega t + 2\phi)].$$

    Since the time-average of the second term on the right is zero, we can express the time-average of this identity as

$$\langle \cos^2(\omega t + \phi) \rangle = \langle \frac{1}{2}[1 + \cos(2\omega t + 2\phi)] \rangle = \frac{1}{2},$$

where the angular brackets denote the time-averaging procedure.



- Consequently, the result

$$\mathbf{E} \times \mathbf{H} = \pm\eta \cos^2(\omega t \mp \beta z)\hat{z}\, \frac{\mathrm{W}}{\mathrm{m}^2}$$

  from Example 1 implies that

$$\langle \mathbf{E} \times \mathbf{H} \rangle = \pm\eta\frac{1}{2}\hat{z}\,\frac{\mathrm{W}}{\mathrm{m}^2} = \pm 60\pi\,\hat{z}\,\frac{\mathrm{W}}{\mathrm{m}^2},$$

  which represent the time-average power per unit area transported by the waves radiated by the current sheet.

- In Poynting theorem the Joule heating term $\mathbf{J} \cdot \mathbf{E}$ is **power absorbed per unit volume**, and, accordingly, $-\mathbf{J} \cdot \mathbf{E}$ is **power injected per unit volume**.

  – Likewise, $\pm\mathbf{J}_s \cdot \mathbf{E}$ can be interpreted as **power absorbed/injected per unit area** on a surface.

  In Example 1 above we calculated an instantaneous injected power density of

$$-\mathbf{J}_s \cdot \mathbf{E} \;=\; 2\eta \cos^2(\omega t)\,\frac{\mathrm{W}}{\mathrm{m}^2}.$$

Clearly, its time-aveage works out as

$$\langle -\mathbf{J}_s \cdot \mathbf{E} \rangle \;=\; \eta\,\frac{\mathrm{W}}{\mathrm{m}^2} = 120\pi\,\frac{\mathrm{W}}{\mathrm{m}^2}.$$

- Note that $\langle -\mathbf{J}_s \cdot \mathbf{E} \rangle$ exactly matches the sum of $|\langle \mathbf{E} \times \mathbf{H} \rangle|$ calculated on both sides of the current sheet, in conformity with energy conservation principle (Poynting theorem).