## 22 Phasor form of Maxwell's equations and damped waves in conducting media

- When the fields and the sources in Maxwell's equations are all monochromatic functions of time expressed in terms of their phasors, Maxwell's equations can be transformed into the phasor domain.
- In the phasor domain all

$$
\frac{\partial}{\partial t} \rightarrow j \omega
$$

and all variables $\mathbf{D}, \rho$, etc. are replaced by their phasors $\tilde{\mathbf{D}}, \tilde{\rho}$, and so on. With those changes Maxwell's equations take the form shown in the margin.

- Also in these equations it is implied that

$$
\begin{aligned}
\tilde{\mathbf{D}} & =\epsilon \tilde{\mathbf{E}} \\
\tilde{\mathbf{B}} & =\mu \tilde{\mathbf{H}} \\
\tilde{\mathbf{J}} & =\sigma \tilde{\mathbf{E}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla \cdot \tilde{\mathbf{D}} & =\tilde{\rho} \\
\nabla \cdot \tilde{\mathbf{B}} & =0 \\
\nabla \times \tilde{\mathbf{E}} & =-j \omega \tilde{\mathbf{B}} \\
\nabla \times \tilde{\mathbf{H}} & =\tilde{\mathbf{J}}+j \omega \tilde{\mathbf{D}}
\end{aligned}
$$

where $\epsilon, \mu$, and $\sigma$ could be a function of frequency $\omega$ (as, strictly speaking, they all are as seen in Lecture 11).

- We can derive from the phasor form Maxwell's equations shown in the margin the TEM wave properties obtained earlier on using the time-domain equations by assuming $\tilde{\rho}=\tilde{\mathbf{J}}=0$.

We will do that, and and after that relax the requirement $\tilde{\mathbf{J}}=0$ with $\tilde{\mathbf{J}}=\sigma \tilde{\mathbf{E}}$ to examine how TEM waves propagate in conducting media.

- With $\tilde{\rho}=\tilde{\mathbf{J}}=0$ the phasor form Maxwell's equation take their simplified forms shown in the margin.
- Using

$$
\nabla \times[\nabla \times \tilde{\mathbf{E}}=-j \omega \mu \tilde{\mathbf{H}}] \Rightarrow-\nabla^{2} \tilde{\mathbf{E}}=-j \omega \mu \nabla \times \tilde{\mathbf{H}}
$$

which combines with the Ampere's law to produce

$$
\begin{aligned}
\nabla \cdot \tilde{\mathbf{E}} & =0 \\
\nabla \cdot \tilde{\mathbf{H}} & =0 \\
\nabla \times \tilde{\mathbf{E}} & =-j \omega \mu \tilde{\mathbf{H}} \\
\nabla \times \tilde{\mathbf{H}} & =j \omega \epsilon \tilde{\mathbf{E}}
\end{aligned}
$$

$$
\nabla^{2} \tilde{\mathbf{E}}+\omega^{2} \mu \epsilon \tilde{\mathbf{E}}=0
$$

- For $x$-polarized waves with phasors

$$
\tilde{\mathbf{E}}=\hat{x} \tilde{E}_{x}(z),
$$

the phasor wave equation above simplifies as

$$
\frac{\partial^{2}}{\partial z^{2}} \tilde{E}_{x}+\omega^{2} \mu \epsilon \tilde{E}_{x}=0
$$

- Try solutions of the form

$$
\tilde{E}_{x}(z)=e^{-\gamma z} \text { or } e^{\gamma z}
$$

where $\gamma$ is to be determined.

- Upon substitution into wave equation both of these lead to

$$
\left(\gamma^{2}+\omega^{2} \mu \epsilon\right) \tilde{E}_{x}=0,
$$

which yields

$$
\gamma^{2}+\omega^{2} \mu \epsilon=0 \Rightarrow \gamma^{2}=-\omega^{2} \mu \epsilon
$$

from which one possibility is

$$
\gamma=j \beta, \quad \text { with } \quad \beta \equiv \omega \sqrt{\mu \epsilon} .
$$

Thus viable phasor solutions are

$$
\tilde{E}_{x}(z)=e^{\mp j \beta z}
$$

as we already knew.

- Furthermore, using the phasor form Faraday's law it is easy to show that

$$
\tilde{H}_{y}= \pm \frac{e^{\mp j \beta z}}{\eta} \text { with } \quad \eta=\sqrt{\frac{\mu}{\omega}} .
$$

Note that we have recovered above the familiar properties of plane TEM waves using phasor methods.
Next, the phasor method carries us to a new domain that cannot be easily examined using time-domain methods.

- With $\tilde{\rho}=0$ but $\tilde{\mathbf{J}}=\sigma \tilde{\mathbf{E}} \neq 0$, implying non-zero conductivity $\sigma$, the pertinent phasor form equations are as shown in the margin.
- This is the same set as before, except that

$$
j \omega \epsilon \text { has been replaced by } \sigma+j \omega \epsilon \text {. }
$$

$$
\begin{aligned}
\nabla \cdot \tilde{\mathbf{E}} & =0 \\
\nabla \cdot \tilde{\mathbf{H}} & =0 \\
\nabla \times \tilde{\mathbf{E}} & =-j \omega \mu \tilde{\mathbf{H}}
\end{aligned}
$$

Thus, we can make use of phasor wave solutions above after ap- $\nabla \times \tilde{\mathbf{H}}=\sigma \tilde{\mathbf{E}}+j \omega \in \tilde{\mathbf{E}}$ plying the following modifications to $\gamma$ and $\eta$ :

$$
=(\sigma+j \omega \epsilon) \tilde{\mathbf{E}}
$$

1. 

$$
\gamma^{2}=-\omega^{2} \mu \epsilon=(j \omega \mu)(j \omega \epsilon) \begin{array}{|c}
\Rightarrow \Rightarrow \\
\sigma \neq 0
\end{array} \quad \gamma=\sqrt{(j \omega \mu)(\sigma+j \omega \epsilon)}
$$

2. 

$$
\left.\eta=\sqrt{\frac{\mu}{\epsilon}}=\sqrt{\frac{j \omega \mu}{j \omega \epsilon}} \quad \begin{array}{c}
\Rightarrow \Rightarrow \\
\sigma \neq 0
\end{array}\right) \eta=\sqrt{\frac{j \omega \mu}{\sigma+j \omega \epsilon}} .
$$

Note that the modified $\gamma$ and $\eta$ satisfy

$$
\gamma \eta=j \omega \mu \text { and } \frac{\gamma}{\eta}=\sigma+j \omega \epsilon \quad \quad \mu=\frac{\gamma \eta}{j \omega}
$$

$$
\begin{aligned}
\mu & =\frac{\gamma \eta}{j \omega} \\
\sigma & =\operatorname{Re}\left\{\frac{\gamma}{\eta}\right\}
\end{aligned}
$$

leading to useful relations shown in the margin (assuming real valued $\sigma$ and $\epsilon$ ).

- In terms of $\gamma$ and $\eta$ above, we can express an $x$-polarized plane wave propagating in $z$ direction in terms of phasors

$$
\tilde{\mathbf{E}}=\hat{x} E_{o} e^{\mp \gamma z} \text { and } \tilde{\mathbf{H}}= \pm \hat{y} \frac{E_{o}}{\eta} e^{\mp \gamma z}
$$

where $E_{o}$ is an arbitrary complex constant (complex wave amplitude).
(a) Damped wave snapshot at $t=0$ together with exponential envelope

(b) Snaphot at $t>0$, with $t=0$ waveform for comparison
$\gamma=\sqrt{(j \omega \mu)(\sigma+j \omega \epsilon)} \equiv \alpha+j \beta$, so that $\alpha=\operatorname{Re}\{\gamma\}$ and $\beta=\operatorname{Im}\{\gamma\}$, and
$\eta=\sqrt{\frac{j \omega \mu}{\sigma+j \omega \epsilon}} \equiv|\eta| e^{j \tau}$ so that $|\eta|=\left|\sqrt{\frac{j \omega \mu}{\sigma+j \omega \epsilon}}\right|$ and $\tau=\angle \sqrt{\frac{j \omega \mu}{\sigma+j \omega \epsilon}}$.
$-\beta$ appears within cosine argument and determines the wavelength

$$
\lambda=\frac{2 \pi}{\beta}
$$

and propagation speed
and, therefore,

$$
\tilde{\mathbf{E}}=\hat{x} E_{o} e^{\mp j \beta z} \text { and } \tilde{\mathbf{H}}= \pm \frac{\hat{y} E_{o} e^{\mp j \beta z}}{\eta}
$$

as before. In this case $\alpha=\tau=0$.

$$
v_{p}=\frac{\omega}{\beta} .
$$

- $\alpha$ controls wave attenuation by

$$
e^{\mp \alpha z}
$$

factor in propagation direction.
2. Another case of imperfect dielectric (or "lousy" conductor) occurs when $\sigma$ is not zero, but it is so small that are justified in using

$$
(1 \pm a)^{p} \approx 1 \pm p a, \text { if }|a| \ll 1
$$

with $p=\frac{1}{2}$ as follows: For $\frac{\sigma}{\omega \epsilon} \ll 1$,
$\gamma=\sqrt{(j \omega \mu)(\sigma+j \omega \epsilon)}=j \omega \sqrt{\mu \epsilon}\left(1-j \frac{\sigma}{\omega \epsilon}\right)^{1 / 2} \approx j \omega \sqrt{\mu \epsilon}\left(1-j \frac{\sigma}{2 \omega \epsilon}\right)=\frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}}+j \omega \sqrt{\mu \epsilon} ;$
hence

$$
\tilde{\mathbf{E}} \approx \hat{x} E_{o} e^{\mp(\alpha+j \beta) z} \text { with } \alpha=\frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}} \text { and } \beta=\omega \sqrt{\mu \epsilon} ;
$$

also in the same case
$\tilde{\mathbf{H}} \approx \pm \frac{\hat{y} E_{o} e^{\mp(\alpha+j \beta) z}}{\eta}$ with $\eta=\sqrt{\frac{\mu}{\epsilon\left(1-j \frac{\sigma}{\omega \epsilon}\right)}} \approx \sqrt{\frac{\mu}{\epsilon}}\left(1+j \frac{\sigma}{2 \omega \epsilon}\right) \approx \sqrt{\frac{\mu}{\epsilon}} e^{j \tan ^{-1} \frac{\sigma}{2 \omega \epsilon}}$,
such that

$$
|\eta| \approx \sqrt{\frac{\mu}{\epsilon}} \text { and } \tau=\angle \eta \approx \frac{\sigma}{2 \omega \epsilon} .
$$

Note: $\gamma$ and $\eta$ both are complex valued, the consequences of which will be examined later on.
3. A third case of good conductor corresponds to $\frac{\sigma}{\omega \epsilon} \gg 1$. In that case,

$$
\gamma=j \omega \sqrt{\mu \epsilon\left(1-j \frac{\sigma}{\omega \epsilon}\right.} \approx \omega \sqrt{j \mu \frac{\sigma}{\omega}}=(1+j) \sqrt{\frac{\omega \mu \sigma}{2}} \text { and } \eta \approx \sqrt{\frac{\mu}{-j \frac{\sigma}{\omega}}}=\sqrt{\frac{j \omega \mu}{\sigma}}=\sqrt{\frac{\omega \mu}{\sigma}} e^{j \pi / 4} .
$$

Hence,
$\alpha \approx \beta \approx \sqrt{\frac{\omega \mu \sigma}{2}}=\sqrt{\pi f \mu \sigma}$ while $|\eta|=\sqrt{\frac{\omega \mu}{\sigma}}$ and $\tau=\angle \eta=45^{\circ}$.
4. Finally, perfect conductor case corresponds to $\sigma \rightarrow \infty$, in which case $\tilde{E}_{x} \rightarrow 0$ as we will show later on. Wave fields cannot exist in perfect conductors.

- Summarizing, in a homogeneous medium with arbitrary but constant $\mu, \epsilon$, and $\sigma$, time-harmonic plane TEM waves are in terms of

$$
\mathbf{E}=\hat{x} \operatorname{Re}\left\{E_{o} e^{\mp(\alpha+j \beta) z} e^{j \omega t}\right\}=\hat{x}\left|E_{o}\right| e^{\mp \alpha z} \cos \left(\omega t \mp \beta z+\angle E_{o}\right)
$$

and accompanying magnetic fields
$\mathbf{H}= \pm \hat{y} \operatorname{Re}\left\{\frac{E_{o}}{\eta} e^{\mp(\alpha+j \beta) z} e^{j \omega t}\right\}= \pm \hat{y} \frac{\left|E_{o}\right|}{|\eta|} e^{\mp \alpha z} \cos \left(\omega t \mp \beta z+\angle E_{o}-\angle \eta\right)$.

## - Propagation velocity

$$
v_{p}=\frac{\omega}{\beta}=\frac{\omega}{\operatorname{Im}\{\sqrt{(j \omega \mu)(\sigma+j \omega \epsilon)}\}}
$$

now depends on frequency $\omega$ and it describes the speed of the nodes (zero-crossings, not modified by the attenuation factor) of the field waveform. Subscript $p$ is introduced to distinguish $v_{p}$ - also called phase velocity - from group velocity $v_{g}$ discussed in ECE 450 (velocity of narrowband wave packets).

(b) Snaphot at $t>0$, with $t=0$ waveform for comparison


- $\beta$ appears within cosine argument and determines the wavelength

$$
\lambda=\frac{2 \pi}{\beta}
$$

and propagation speed

$$
v_{p}=\frac{\omega}{\beta} .
$$

- $\alpha$ controls wave attenuation by

$$
e^{\mp \alpha z}
$$

factor in propagation direction.

- Wavelength

$$
\lambda=\frac{2 \pi}{\beta}=\frac{v_{p}}{f}
$$

now depends on frequency $f$ via both the numerator and the denominator, and measures twice the distance between successive nodes of the waveform.

- Penetration depth (also called skin depth if very small)

$$
\delta \equiv \frac{1}{\alpha}=\frac{1}{\operatorname{Re}\{\sqrt{(j \omega \mu)(\sigma+j \omega \epsilon)}\}}
$$

is the distance for the field strength to be reduced by $e^{-1}$ factor in its direction of propagation.

- For a fixed $\sigma$, and a sufficiently large $\omega$, the penetration depth

$$
\delta \approx \frac{2}{\sigma \sqrt{\frac{\mu}{\epsilon}}} \text { Imperfect dielectric formula }
$$

which can be very small if $\sigma$ is large - with small $\delta$ the wave is severely attenuated as it propagates.

- For a fixed $\sigma$, and a sufficiently small $\omega$,

$$
\delta \approx \sqrt{\frac{2}{\mu \omega \sigma}}=\frac{1}{\sqrt{\pi f \mu \sigma}} \text { Good conductor "skin depth" formula }
$$

which, although small with large $\sigma$, increases as $\omega$ decreases, making low frequencies to be preferable in applications requiring propagating through lossy media with large $\sigma$, such as in sea-water.

(b) Snaphot at $t>0$, with $t=0$ waveform for comparison

$-\beta$ appears within cosine argument and determines the wavelength

$$
\lambda=\frac{2 \pi}{\beta}
$$

and propagation speed

$$
v_{p}=\frac{\omega}{\beta} .
$$

- $\alpha$ controls wave attenuation by

$$
e^{\mp \alpha z}
$$

factor in propagation direction.

