## 22 Phasor form of Maxwell's equations and damped waves in conducting media

- When the fields and the sources in Maxwell's equations are all monochromatic functions of time expressed in terms of their phasors, Maxwell's equations can be transformed into the phasor domain.
  - In the phasor domain all

$$\frac{\partial}{\partial t} \to j\omega$$

 $\nabla \cdot \mathbf{D} = \rho$   $\nabla \cdot \mathbf{B} = 0$   $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$  $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$ 

and all variables  $\mathbf{D}$ ,  $\rho$ , etc. are replaced by their phasors  $\tilde{\mathbf{D}}$ ,  $\tilde{\rho}$ , and so on. With those changes Maxwell's equations take the form shown in the margin.

- Also in these equations it is implied that

$$\begin{split} \tilde{\mathbf{D}} &= \epsilon \tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} &= \mu \tilde{\mathbf{H}} \\ \tilde{\mathbf{J}} &= \sigma \tilde{\mathbf{E}} \end{split}$$

 $\begin{aligned} \nabla \cdot \tilde{\mathbf{D}} &= \tilde{\rho} \\ \nabla \cdot \tilde{\mathbf{B}} &= 0 \\ \nabla \times \tilde{\mathbf{E}} &= -j\omega \tilde{\mathbf{B}} \\ \nabla \times \tilde{\mathbf{H}} &= \tilde{\mathbf{J}} + j\omega \tilde{\mathbf{D}} \end{aligned}$ 

where  $\epsilon$ ,  $\mu$ , and  $\sigma$  could be a function of frequency  $\omega$  (as, strictly speaking, they all are as seen in Lecture 11).

– We can derive from the phasor form Maxwell's equations shown in the margin the TEM wave properties obtained earlier on using the time-domain equations by assuming  $\tilde{\rho} = \tilde{\mathbf{J}} = 0$ . We will do that, and and after that relax the requirement  $\tilde{\mathbf{J}} = 0$  with  $\tilde{\mathbf{J}} = \sigma \tilde{\mathbf{E}}$  to examine how TEM waves propagate in conducting media.

• With  $\tilde{\rho} = \tilde{\mathbf{J}} = 0$  the phasor form Maxwell's equation take their simplified forms shown in the margin.

- Using

$$\nabla \times [\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}] \ \Rightarrow \ -\nabla^2\tilde{\mathbf{E}} = -j\omega\mu\nabla \times \tilde{\mathbf{H}}$$

which combines with the Ampere's law to produce

$$\nabla^2 \tilde{\mathbf{E}} + \omega^2 \mu \epsilon \tilde{\mathbf{E}} = 0.$$

- For x-polarized waves with phasors

 $\tilde{\mathbf{E}} = \hat{x}\tilde{E}_x(z),$ 

the phasor wave equation above simplifies as

$$\frac{\partial^2}{\partial z^2}\tilde{E}_x + \omega^2 \mu \epsilon \tilde{E}_x = 0.$$

- Try solutions of the form

$$\tilde{E}_x(z) = e^{-\gamma z}$$
 or  $e^{\gamma z}$ 

where  $\gamma$  is to be determined.

 $\nabla \cdot \tilde{\mathbf{E}} = 0$   $\nabla \cdot \tilde{\mathbf{H}} = 0$   $\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}$  $\nabla \times \tilde{\mathbf{H}} = j\omega\epsilon\tilde{\mathbf{E}}$  - Upon substitution into wave equation both of these lead to

$$(\gamma^2 + \omega^2 \mu \epsilon) \tilde{E}_x = 0,$$

which yields

$$\gamma^2 + \omega^2 \mu \epsilon = 0 \quad \Rightarrow \quad \gamma^2 = -\omega^2 \mu \epsilon$$

from which one possibility is

$$\gamma = j\beta$$
, with  $\beta \equiv \omega \sqrt{\mu\epsilon}$ .

Thus viable phasor solutions are

$$\tilde{E}_x(z) = e^{\mp j\beta z}$$

as we already knew.

- Furthermore, using the phasor form Faraday's law it is easy to show that

$$\tilde{H}_y = \pm \frac{e^{\pm j\beta z}}{\eta}$$
 with  $\eta = \sqrt{\frac{\mu}{\omega}}$ .

Note that we have recovered above the familiar properties of plane TEM waves using phasor methods.

Next, the phasor method carries us to a new domain that cannot be easily examined using time-domain methods.

- With  $\tilde{\rho} = 0$  but  $\tilde{\mathbf{J}} = \sigma \tilde{\mathbf{E}} \neq 0$ , implying non-zero conductivity  $\sigma$ , the pertinent phasor form equations are as shown in the margin.
  - This is the same set as before, except that  $\begin{aligned} \nabla \cdot \tilde{\mathbf{E}} &= 0 \\ \nabla \cdot \tilde{\mathbf{H}} &= 0 \\ \nabla \times \tilde{\mathbf{E}} &= -j\omega\mu\tilde{\mathbf{H}} \end{aligned}$ Thus, we can make use of phasor wave solutions above after approximation  $\nabla \times \tilde{\mathbf{E}} &= -j\omega\mu\tilde{\mathbf{H}} \\ \nabla \times \tilde{\mathbf{E}} &= -j\omega\mu\tilde{\mathbf{H}} \end{aligned}$ Thus, we can make use of phasor wave solutions above after approximation  $\nabla \times \tilde{\mathbf{H}} &= \sigma\tilde{\mathbf{E}} + j\omega\epsilon\tilde{\mathbf{E}} \\ plying the following modifications to <math>\gamma$  and  $\eta$ :  $\begin{aligned} \nabla \cdot \tilde{\mathbf{E}} &= 0 \\ \nabla \times \tilde{\mathbf{E}} &= -j\omega\mu\tilde{\mathbf{H}} \\ \nabla \times \tilde{\mathbf{H}} &= \sigma\tilde{\mathbf{E}} + j\omega\epsilon\tilde{\mathbf{E}} \\ &= (\sigma + j\omega\epsilon)\tilde{\mathbf{E}} \end{aligned}$

1.

$$\gamma^2 = -\omega^2 \mu \epsilon = (j\omega\mu)(j\omega\epsilon) \quad \Rightarrow \Rightarrow \\ \sigma \neq 0 \quad \gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\epsilon)}$$

2.

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\epsilon}} \quad \Rightarrow \Rightarrow \\ \sigma \neq 0 \quad \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}.$$

Note that the modified  $\gamma$  and  $\eta$  satisfy

$$\gamma \eta = j\omega \mu$$
 and  $\frac{\gamma}{\eta} = \sigma + j\omega \epsilon$   $\mu = \frac{\gamma \eta}{j\omega}$ 

leading to useful relations shown in the margin (assuming real valued  $\sigma$  and  $\epsilon$ ).

$$\mu = \frac{\gamma \eta}{j\omega}$$
  
$$\sigma = \operatorname{Re}\left\{\frac{\gamma}{\eta}\right\}$$
  
$$\epsilon = \frac{1}{\omega}\operatorname{Im}\left\{\frac{\gamma}{\eta}\right\}$$

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• In terms of  $\gamma$  and  $\eta$  above, we can express an x-polarized plane wave propagating in z direction in terms of phasors

$$\tilde{\mathbf{E}} = \hat{x} E_o e^{\mp \gamma z}$$
 and  $\tilde{\mathbf{H}} = \pm \hat{y} \frac{E_o}{\eta} e^{\mp \gamma z}$ 

where  $E_o$  is an arbitrary complex constant (complex wave amplitude).

• In expanded forms  $\gamma$  and  $\eta$  appear as:

$$\gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\epsilon)} \equiv \alpha + j\beta$$
, so that  $\alpha = \operatorname{Re}\{\gamma\}$  and  $\beta = \operatorname{Im}\{\gamma\}$ ,

and

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \equiv |\eta|e^{j\tau} \text{ so that } |\eta| = |\sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}| \text{ and } \tau = \angle \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}.$$

1. In the special case of a **perfect dielectric** with  $\sigma = 0$ , we find

$$\gamma = j\omega\sqrt{\mu\epsilon} \equiv j\beta$$
 and  $\eta = \sqrt{\frac{\mu}{\epsilon}}$ ,

and, therefore,

$$\tilde{\mathbf{E}} = \hat{x} E_o e^{\mp j\beta z}$$
 and  $\tilde{\mathbf{H}} = \pm \frac{\hat{y} E_o e^{\mp j\beta z}}{\eta}$ 

as before. In this case  $\alpha = \tau = 0$ .

 $\beta$  appears within cosine argument and determines the wavelength

$$\lambda = \frac{2\pi}{\beta}$$

and propagation speed

$$v_p = \frac{\omega}{\beta}.$$

 $- \alpha$  controls wave attenuation by

 $e^{\mp \alpha z}$ 

factor in propagation direction.

(a) Damped wave snapshot at t=0 together with exponential envelope



(b) Snaphot at t>0, with t=0 waveform

for comparison

2. Another case of **imperfect dielectric** (or "lousy" conductor) occurs when  $\sigma$  is not zero, but it is so small that are justified in using

$$(1\pm a)^p \approx 1\pm pa$$
, if  $|a|\ll 1$ ,

with  $p = \frac{1}{2}$  as follows: For  $\frac{\sigma}{\omega\epsilon} \ll 1$ ,

$$\gamma = \sqrt{(j\omega\mu)(\sigma + j\omega\epsilon)} = j\omega\sqrt{\mu\epsilon}(1 - j\frac{\sigma}{\omega\epsilon})^{1/2} \approx j\omega\sqrt{\mu\epsilon}(1 - j\frac{\sigma}{2\omega\epsilon}) = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} + j\omega\sqrt{\mu\epsilon};$$

hence

$$\tilde{\mathbf{E}} \approx \hat{x} E_o e^{\mp (\alpha + j\beta)z}$$
 with  $\alpha = \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}}$  and  $\beta = \omega \sqrt{\mu\epsilon}$ ;

also in the same case

$$\tilde{\mathbf{H}} \approx \pm \frac{\hat{y} E_o e^{\mp (\alpha + j\beta)z}}{\eta} \text{ with } \eta = \sqrt{\frac{\mu}{\epsilon(1 - j\frac{\sigma}{\omega\epsilon})}} \approx \sqrt{\frac{\mu}{\epsilon}} (1 + j\frac{\sigma}{2\omega\epsilon}) \approx \sqrt{\frac{\mu}{\epsilon}} e^{j\tan^{-1}\frac{\sigma}{2\omega\epsilon}},$$

such that

$$|\eta| \approx \sqrt{\frac{\mu}{\epsilon}} \text{ and } \tau = \angle \eta \approx \frac{\sigma}{2\omega\epsilon}.$$

Note:  $\gamma$  and  $\eta$  both are *complex* valued, the consequences of which will be examined later on.

3. A third case of **good conductor** corresponds to  $\frac{\sigma}{\omega\epsilon} \gg 1$ . In that case,

$$\gamma = j\omega\sqrt{\mu\epsilon(1-j\frac{\sigma}{\omega\epsilon})} \approx \omega\sqrt{j\mu\frac{\sigma}{\omega}} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}} \text{ and } \eta \approx \sqrt{\frac{\mu}{-j\frac{\sigma}{\omega}}} = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}e^{j\pi/4}$$

Hence,

$$\alpha \approx \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}$$
 while  $|\eta| = \sqrt{\frac{\omega\mu}{\sigma}}$  and  $\tau = \angle \eta = 45^{\circ}$ .

- 4. Finally, **perfect conductor** case corresponds to  $\sigma \to \infty$ , in which case  $\tilde{E}_x \to 0$  as we will show later on. Wave fields cannot exist in perfect conductors.
- Summarizing, in a homogeneous medium with arbitrary but constant  $\mu$ ,  $\epsilon$ , and  $\sigma$ , time-harmonic plane TEM waves are in terms of

$$\mathbf{E} = \hat{x} \operatorname{Re} \{ E_o e^{\mp (\alpha + j\beta)z} e^{j\omega t} \} = \hat{x} | E_o | e^{\mp \alpha z} \cos(\omega t \mp \beta z + \angle E_o) \}$$

and accompanying magnetic fields

$$\mathbf{H} = \pm \hat{y} \operatorname{Re} \{ \frac{E_o}{\eta} e^{\mp (\alpha + j\beta)z} e^{j\omega t} \} = \pm \hat{y} \frac{|E_o|}{|\eta|} e^{\mp \alpha z} \cos(\omega t \mp \beta z + \angle E_o - \angle \eta).$$

• Propagation velocity

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\operatorname{Im}\{\sqrt{(j\omega\mu)(\sigma + j\omega\epsilon)}\}},$$

now depends on frequency  $\omega$  and it describes the speed of the **nodes** (zero-crossings, not modified by the attenuation factor) of the field waveform. Subscript p is introduced to distinguish  $v_p$  — also called *phase velocity* — from *group velocity*  $v_g$  discussed in ECE 450 (velocity of narrowband wave packets).

(a) Damped wave snapshot at t=0 together with exponential envelope





 β appears within cosine argument and determines the wavelength

$$\lambda = \frac{2\pi}{\beta}$$

and propagation speed

$$v_p = \frac{\omega}{\beta}.$$

•  $\alpha$  controls wave attenuation by

 $e^{\mp \alpha z}$ 

factor in propagation direction.

• Wavelength

$$\lambda = \frac{2\pi}{\beta} = \frac{v_p}{f}$$

now depends on frequency f via both the numerator and the denominator, and measures twice the distance between successive nodes of the waveform.

• **Penetration depth** (also called *skin depth* if very small)

$$\delta \equiv \frac{1}{\alpha} = \frac{1}{\operatorname{Re}\{\sqrt{(j\omega\mu)(\sigma + j\omega\epsilon)}\}}$$

is the distance for the field strength to be reduced by  $e^{-1}$  factor in its direction of propagation.

- For a fixed  $\sigma$ , and a sufficiently large  $\omega$ , the penetration depth

$$\delta \approx \frac{2}{\sigma \sqrt{\frac{\mu}{\epsilon}}}$$
 Imperfect dielectric formula

which can be very small if  $\sigma$  is large — with small  $\delta$  the wave is severely attenuated as it propagates.

– For a fixed  $\sigma$ , and a sufficiently small  $\omega$ ,

$$\delta \approx \sqrt{\frac{2}{\mu\omega\sigma}} = \frac{1}{\sqrt{\pi f\mu\sigma}}$$
 Good conductor "skin depth" formula

which, although small with large  $\sigma$ , increases as  $\omega$  decreases, making *low frequencies to be preferable* in applications requiring propagating through lossy media with large  $\sigma$ , such as in sea-water.

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(a) Damped wave snapshot at t=0 together with exponential envelope





 $\beta$  appears within cosine argument and determines the wavelength

$$\lambda = \frac{2\pi}{\beta}$$

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$$v_p = \frac{\omega}{\beta}.$$

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