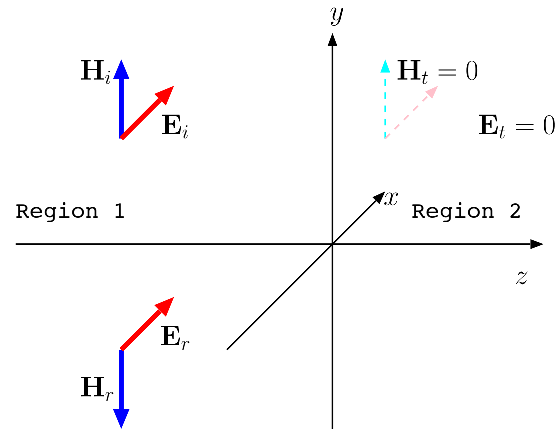
26 Standing waves, radiation pressure



We continue in this lecture with our studies of wave reflection and transmission at a plane boundary between two homogeneous media.

- In case of *total reflection* from a perfectly conducting mirror placed at $z = 0$ surface, $\Gamma = -1$, and the incident and reflected waves in $z < 0$ region combine to produce **standing waves** of electric and magnetic field:

  - **Incident wave** (a traveling wave going in z-direction):

$$\tilde{\mathbf{E}}_i = \hat{x}E_o e^{-j\beta_1 z} \quad \text{and} \quad \tilde{\mathbf{H}}_i = \hat{y}\frac{E_o}{\eta_1}e^{-j\beta_1 z},$$

  - **Reflected wave** (a traveling wave going in $-z$-direction):

$$\tilde{\mathbf{E}}_r = -\hat{x}E_o e^{j\beta_1 z} \quad \text{and} \quad \tilde{\mathbf{H}}_r = \hat{y}\frac{E_o}{\eta_1}e^{j\beta_1 z},$$

  - **Standing wave**:

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_i + \tilde{\mathbf{E}}_r = \hat{x}E_o(e^{-j\beta_1 z} - e^{j\beta_1 z}) \quad \text{and} \quad \tilde{\mathbf{H}} = \tilde{\mathbf{H}}_i + \tilde{\mathbf{H}}_r = \hat{y}\frac{E_o}{\eta_1}(e^{-j\beta_1 z} + e^{j\beta_1 z})$$

which simplify as

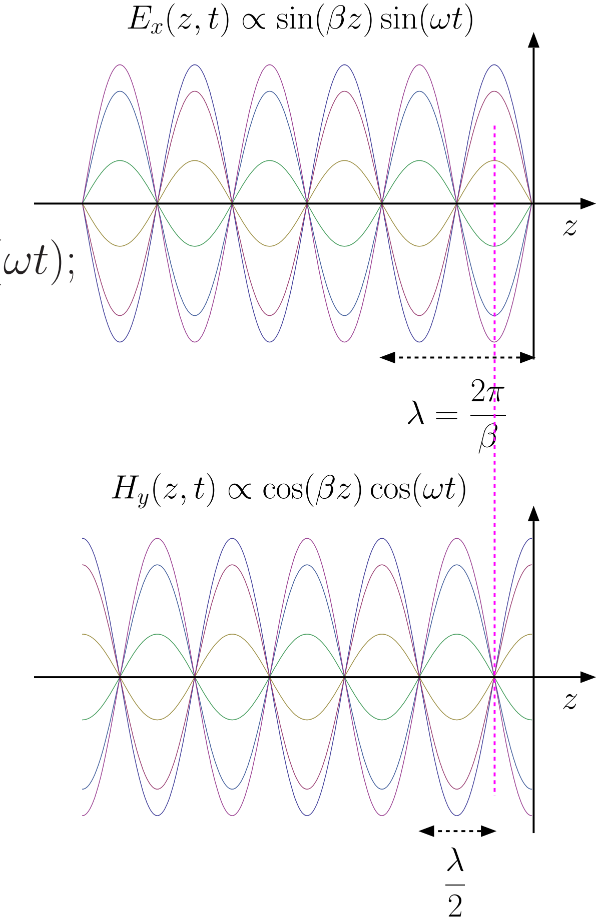
$$\tilde{\mathbf{E}} = -j\hat{x}2E_o\sin(\beta_1 z) \quad \text{and} \quad \tilde{\mathbf{H}} = \hat{y}\frac{2E_o}{\eta_1}\cos(\beta_1 z).$$

Standing waves

These are called ***standing wave*** phasors because when we go to the time-domain (by multiplying with $e^{j\omega t}$ and taking the real time as usual) we obtain:

$$\mathbf{E}(z,t) = \hat{x}2E_o \sin(\beta_1 z)\sin(\omega t) \quad \text{and} \quad \mathbf{H}(z,t) = \hat{y}\frac{2E_o}{\eta_1}\cos(\beta_1 z)\cos(\omega t);$$

these, unlike d'Alembert solutions of the format $f(t \mp \frac{z}{v})$, describe oscillations in time t, with different amplitudes at different positions z (see margin and the animation linked in class calendar).



Note: Nulls in Ex and Hy are separated by half wavelength.

Adjacent nulls of Ex and Hy are separated by quarter wavelength.

It is useful to think of nulls of Ex as "shorts" in analogy to shorts in circuits where v=0.

Conductor shorts Ex on its surface where a current flows.

Also useful to think of nulls of Hy as "opens" in analogy to opens in circuits where i=0.

– Standing waves carry no net energy, that is, with standing wave fields we have

$$\langle \mathbf{E} \times \mathbf{H} \rangle = 0,$$

because of the cancellation of the power transported by its traveling wave components in opposite directions.

Verification: Using the phasors

$$\tilde{\mathbf{E}} = -j\hat{x}2E_o \sin(\beta_1 z) \quad \text{and} \quad \tilde{\mathbf{H}} = \hat{y}\frac{2E_o}{\eta_1}\cos(\beta_1 z),$$

we have

$$\begin{aligned}\langle \mathbf{E} \times \mathbf{H} \rangle &= \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\} = \frac{1}{2}\mathrm{Re}\{-j\hat{x}2E_o\sin(\beta_1 z) \times \hat{y}\frac{2E_o}{\eta_1}\cos(\beta_1 z)\} \\ &= \hat{z}\frac{2E_o^2}{\eta_1}\sin(\beta_1 z)\cos(\beta_1 z)\mathrm{Re}\{-j\} = 0.\end{aligned}$$

– Note that $\mathbf{E}(0,t) = 0$ on $z = 0$ surface satisfying the tangential electric boundary condition as expected (recall that the fields are zero within the perfect conducting mirror).

– Also note that

$$\mathbf{H}(0,t) = \hat{y}\frac{2E_o}{\eta_1}\cos(\omega t)$$

on $z = 0$ surface. Since this tangential magnetic field is not zero, boundary condition equations imply that there must be an oscillating surface current

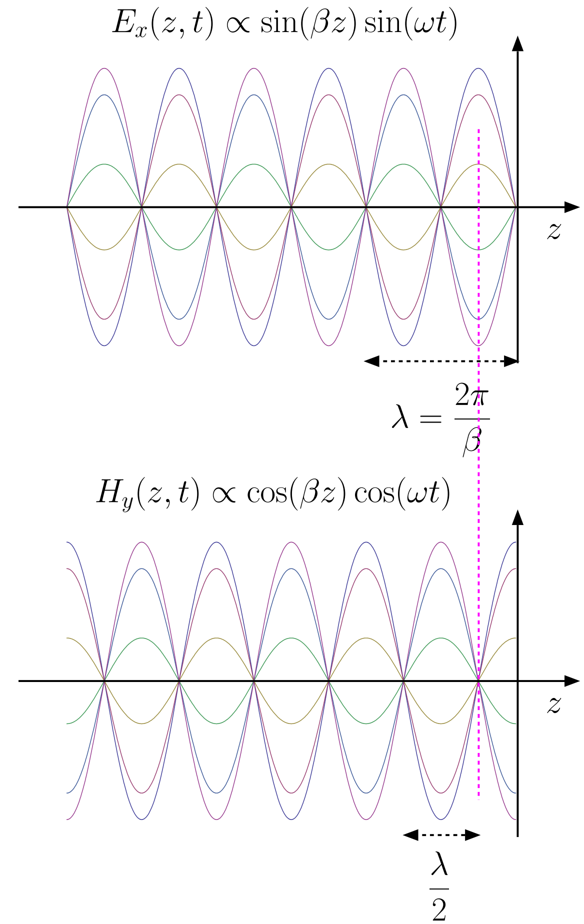
$$\mathbf{J}_s = \hat{x}\frac{2E_o}{\eta_1}\cos(\omega t)\ \frac{\mathrm{A}}{\mathrm{m}},$$

satisfying

$$-\hat{z} \times \mathbf{H}(0,t) = \mathbf{J}_s.$$

$\mathbf{J}_s$ on mirror surfaces is really a convenient *idealization* of volume currents flowing in thin layers — just a few skin depths — near good-conductor surfaces (real-life mirrors are good but not perfect conductors!). Radiation due to $\mathbf{J}_s$ *in effect* causes the reflected wave and also cancels out the incident wave field in $z > 0$.

Next we examine reflections from a good conductor and see of how the limiting case of a perfect conductor is naturally reached.



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Conductor shorts Ex on its surface where a current flows.

Also useful to think of nulls of Hy as "opens" in analogy to opens in circuits where i=0.

- Going back to the partial reflection case, consider the transmitted fields $\tilde{\mathbf{E}}_t$ and $\tilde{\mathbf{H}}_t$ in Region 2 shown in the margin. Also shown in the margin are the phasors for current density $\mathbf{J}_t$ and magnetic flux density $\mathbf{B}_t$.

$$\tilde{\mathbf{E}}_t = \hat{x}\tau E_o e^{-\gamma_2 z}$$

$$\tilde{\mathbf{H}}_t = \hat{y}\frac{\tau E_o}{\eta_2} e^{-\gamma_2 z}$$

$$\tilde{\mathbf{J}}_t = \sigma_2 \tilde{\mathbf{E}}_t = \hat{x}\sigma_2 \tau E_o e^{-\gamma_2 z}$$

$$\tilde{\mathbf{B}}_t = \mu_2 \tilde{\mathbf{H}}_t = \hat{y}\frac{\mu_2 \tau E_o}{\eta_2} e^{-\gamma_2 z}$$

In the box below we integrate the volumetric current density $\mathbf{J}_t$ of a good conductor from $z = 0$ to ∞ and find out that this "depth integral" matches the surface current density found above for the case of perfect conductor. In this calculation we assume that Region 1 is vacuum, and also take $\mu_2 = \mu_o$:

Effective surface current: Assuming that Region 2 is a good conductor,

$$\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\epsilon_2}} \approx \sqrt{\frac{j\omega\mu_2}{\sigma_2}} \quad \text{and} \quad \gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\epsilon_2)} \approx \sqrt{j\omega\mu_2\sigma_2}$$

and therefore

$$\tau = \frac{2\eta_2}{\eta_o + \eta_2} \approx \frac{2\eta_2}{\eta_o} \quad \text{and} \quad \sigma_2\tau \approx \frac{2\sigma_2\eta_2}{\eta_o} = \frac{2\sqrt{j\omega\mu_2\sigma_2}}{\eta_o} = \frac{2\gamma_2}{\eta_o}.$$

The depth integral of the volumetric current density in Region 2, that is, the **effective surface current** of the region is then

$$\int_0^\infty \tilde{\mathbf{J}}_t dz = \hat{x}\int_0^\infty \sigma_2 \tau E_o e^{-\gamma_2 z} dz = \hat{x}E_o \frac{1}{\gamma_2}(\sigma_2\tau) = \hat{x}\frac{2E_o}{\eta_o}$$

in phasor form, matching the phasor of the time-domain result from above, namely

$$\mathbf{J}_s = \hat{x}\frac{2E_o}{\eta_o}\cos(\omega t)\ \frac{\mathrm{A}}{\mathrm{m}}$$

representing the surface current on an idealized perfect conductor surface.

Surface resistance: Let $\tilde{\mathbf{J}}_s$ stand for the effective surface current of a good conductor with a propagation constant

$$\gamma \approx \sqrt{j\omega\mu\sigma} = \alpha + j\beta = \alpha + j\alpha$$

and a volumetric current density $\tilde{\mathbf{J}}(z)$ such that

$$\tilde{\mathbf{J}}_s = \int_{z=0}^{\infty} \tilde{\mathbf{J}}(z) dz = \int_{z=0}^{\infty} \tilde{\mathbf{J}}(0) e^{-\gamma z} dz = \frac{\tilde{\mathbf{J}}(0)}{\gamma}.$$

In that case

$$\tilde{\mathbf{J}}(z) = \tilde{\mathbf{J}}_s \gamma e^{-\gamma z} \quad \text{and} \quad \mathbf{E}(z) = \frac{\tilde{\mathbf{J}}_s \gamma}{\sigma} e^{-\gamma z}$$

inside the good conductor in terms of the effective surface current $\tilde{\mathbf{J}}_s$, and the average power dissipated per unit volume (Joule heating) is

$$\langle \mathbf{J}(z) \cdot \mathbf{E}(z) \rangle = \frac{1}{2} |\tilde{\mathbf{J}}_s \gamma|^2 \frac{e^{-2\alpha z}}{\sigma} = \frac{1}{2} |\tilde{\mathbf{J}}_s|^2 \frac{2\alpha^2 e^{-2\alpha z}}{\sigma}.$$

The depth integral of the same quantity, that is the **power dissipated per unit area**, is then

$$\int_0^{\infty} \langle \mathbf{J}(z) \cdot \mathbf{E}(z) \rangle dz = \frac{1}{2} R_s |\tilde{\mathbf{J}}_s|^2,$$

with

$$R_s \equiv \frac{\alpha}{\sigma} = \frac{\sqrt{\pi f \mu \sigma}}{\sigma} = \sqrt{\frac{\pi f \mu}{\sigma}} \ \ (\Omega)$$

called the **surface resistance**.

The surface resistance concept is useful to model loss effects in waveguides and cavity resonators as studied in ECE 450. Also, we can make use of surface resistance when modeling lossy transmission lines (see Lecture 39).

- Let's finally calculate the magnetic component of the Lorentz force on charge carriers of a good conductor due to the penetrating fields:

Radiation pressure: If there are N free charge carriers per unit volume inside a reflecting mirror, then

$$N\mathbf{F} = Nq\mathbf{v} \times \mathbf{B}_t = \mathbf{J}_t \times \mathbf{B}_t$$

will be the **force per unit volume** of the mirror, expressed in terms of current density $\mathbf{J}_t = Nq\mathbf{v}$ and the magnetic flux density $\mathbf{B}_t$.

Its integral over all z can be interpreted as the total **force per unit area** of the mirror,

$$\mathbf{P}_{rad} = \int_0^\infty \mathbf{J}_t \times \mathbf{B}_t \, dz,$$

having a magnitude known as **radiation pressure** of the reflecting wave. This is a time-varying quantity, with a time-average

$$\begin{aligned}
\langle \mathbf{P}_{rad} \rangle &= \int_0^\infty \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{J}} \times \tilde{\mathbf{B}}^*\} dz \\
&= \hat{z} \int_0^\infty \frac{1}{2}\mathrm{Re}\{(\sigma_2 \tau E_o)(\frac{\mu_2 \tau E_o}{\eta_2})\} e^{-2\alpha_2 z} dz \\
&= \hat{z}\frac{|E_o|^2}{2}\mathrm{Re}\{(\frac{2\gamma_2}{\eta_o})(\frac{\mu_2}{\eta_2}\frac{2\eta_2}{\eta_o})\}\frac{1}{2\alpha_2} = \hat{z}2\frac{|E_o|^2}{2\eta_o}\frac{\mathrm{Re}\{\gamma_2\}}{\alpha_2}\frac{\mu_2}{\eta_o} \\
&= \hat{z}2\frac{|E_o|^2}{2\eta_o}\frac{\mu_o}{\eta_o} = 2\langle \mathbf{S}_i \rangle / c,
\end{aligned}$$

where

$$\langle \mathbf{S}_i \rangle \equiv \hat{z}\frac{|E_o|^2}{2\eta_o}$$

is the time-average Poynting vector of the incident wave reflected from the mirror (factor of 2 in $\langle \mathbf{P}_{rad} \rangle$ is due to the recoil of the wave off the mirror; see Rothman and Boughn, *Am. J. Phys.*, *77*, 122, 1977).

***Radiation pressure* proportional to**

$$\langle \mathbf{S}_i \rangle / c$$

shows that plane-TEM waves not only carry and transport energy, but also momentum.

TEM waves not only *heat*, but also *push*!

((It can also be shown that the momentum density of the wave is

$$\langle \mathbf{S}_i \rangle / c^2 \text{ N.s/m}^3$$

and (spin) angular momentum density

$$\pm \langle \mathbf{S}_i \rangle / \omega c \text{ N.s/m}^2$$

for right- and left-circular waves. Momentum per photon of energy $\hbar\omega$ can be obtained by dividing the above expressions by $|\langle \mathbf{S}_i \rangle| / \omega c \hbar$, the number density of photons in the wave field.))