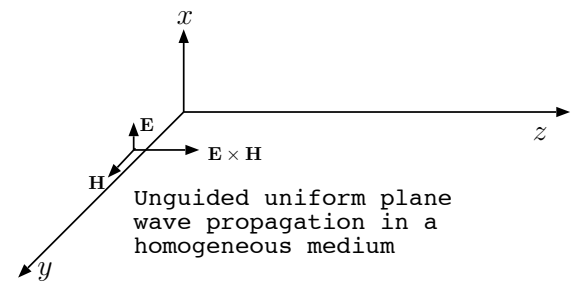
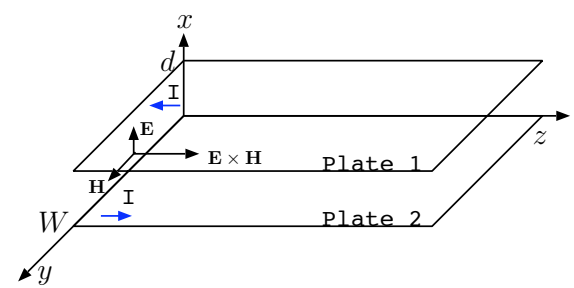
# 27 Guided TEM waves on TL systems




- An $\hat{x}$ polarized plane TEM wave propagating in $z$ direction is depicted in the margin.
  - A pair of conducting plates placed at $x = 0$ and $x = d$ would not perturb the fields except that charge and current density variations would be induced on plate surfaces at $x = 0$ and $x = d$ (on both sides) to satisfy Maxwell's boundary condition equations.




- If charge and currents were confined only to interior surfaces of the plates facing one another, fields **E** and **H** accompanying them would be restricted to the region in between the plates, constituting what we would call **guided waves**.
  - Such a guided wave field confined to the region between the plates will satisfy Maxwell's equations including a minor fringing component that can be neglected when the plate width $W$ is much larger than plate separation $d$.

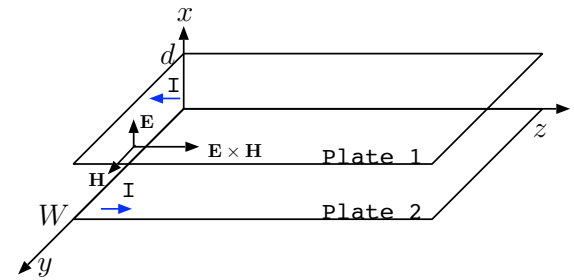
In the following discussion of guided waves in parallel-plate **transmission lines** (TL) we will assume $W \gg d$ and neglect the effects of fringing fields.

- Guided waves produce wavelike surface charge and current variations on plate surfaces.
- Conversely, wavelike charge and current variations on plate surfaces would produce guided wave fields.

It is sufficient to apply a time-varying current and/or charge density at some location $z$ on a parallel-plate TL — e.g., by a time-varying voltage or current source — in order to "excite" the TL with propagating guided fields.

How such excitations propagate away from their "source points" on TL systems will be our main subject of study for the rest of the semester.

- In a parallel-plate TL we ignore any fringing fields and assume that TEM wave fields




$$\mathbf{E} = \hat{x}E_x(z,t) \quad \text{and} \quad \mathbf{H} = \hat{y}H_y(z,t)$$

occupy the region between the plates. For these fields uniform in $x$ and $y$, Faraday's and Ampere's laws reduce to scalar expressions

$$\nabla \times \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t} \quad \Rightarrow \quad \frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t}$$

and

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t} \quad \Rightarrow \quad -\frac{\partial H_y}{\partial z} = \sigma E_x + \epsilon\frac{\partial E_x}{\partial t}.$$

- Now, multiply both equations by $d$ and let

$$V \equiv E_x d \quad \text{voltage drop from plate 2 to plate 1}$$

to obtain

$$\frac{\partial V}{\partial z} = -\mu d\frac{\partial H_y}{\partial t} \quad \text{and} \quad -d\frac{\partial H_y}{\partial z} = \epsilon\frac{\partial V}{\partial t} + \sigma V.$$

- Next, multiply these with $W$ and let

$$I \equiv H_y W \quad \text{current in } z\text{-direction on plate 2}$$

(because $J_{sz} = H_y$ on plate 2) to obtain

$$W\frac{\partial V}{\partial z} = -\mu d\frac{\partial I}{\partial t} \quad \text{and} \quad -d\frac{\partial I}{\partial z} = \epsilon W\frac{\partial V}{\partial t} + \sigma W V.$$

**Note that voltage drop**

$$V = \int_2^1 \mathbf{E}\cdot d\mathbf{l} = E_x d$$

**is uniquely defined — *independent of integration path* — on constant $z$ surfaces because with TEM fields**

$$B_z = \mu H_z = 0,$$

**and consequently circulation**

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} = 0$$

**when $C$ is on constant $z$ plane and $d\mathbf{S} = \pm dxdy\hat{z}$.**

- We can re-write these equations as

$$-\frac{\partial V}{\partial z} = \mathcal{L}\frac{\partial I}{\partial t} \quad \text{and} \quad -\frac{\partial I}{\partial z} = \mathcal{C}\frac{\partial V}{\partial t} + \mathcal{G}V$$

utilizing

$$\mathcal{L} = \mu\frac{d}{W}, \quad \mathcal{C} = \epsilon\frac{W}{d}, \quad \mathcal{G} = \sigma\frac{W}{d}$$

appropriate for the parallel-plate TL — we recognize these parameters as inductance, capacitance, and conductance of the parallel plate TL.

**Telegrapher's equations:**

$$-\frac{\partial V}{\partial z} = \mathcal{L}\frac{\partial I}{\partial t}$$
$$-\frac{\partial I}{\partial z} = \mathcal{C}\frac{\partial V}{\partial t}$$

  - In the equations above the $\mathcal{G}V$ term accounts for Ohmic losses of wave fields having to do with currents leaking between the wires (plates) of the TL.
  - Another possible loss term that we have not picked up — because we assumed infinite conducting plates — is a missing $\mathcal{R}I$ term in the right-hand-side of the first equation.

Rather than correcting for that at this stage, we will drop the $\mathcal{G}V$ term from the second equation, and focus our attention *for a while* (until the last day of the semester, in fact) on *ideal* lossless transmission lines governed by the equations shown in the margin — they are known as known as **telegrapher's equations**[1].

[1] Telegrapher's equations were first compiled by Oliver Heaviside (of close-up method, unit-step, and countless other contributions) in 1880's. Telegraphy was being used worldwide by 1850's as a means of rapid communications.

- Except for $-\frac{\partial}{\partial z}$ on the left, the telegrapher's equations look like the $V - I$ relations of inductors and capacitors (which is the best way of remembering them).

- The equations can be readily combined to obtain a 1D scalar wave equation

$$\frac{\partial^2 V}{\partial z^2} = \mathcal{L}\mathcal{C}\frac{\partial^2 V}{\partial t^2}.$$

In analogy to

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon\frac{\partial^2 E_x}{\partial t^2},$$

the wave equation for $V$ has d'Alembert wave solutions

$$V(z,t) = f(t \mp \frac{z}{v}) \quad \text{where} \quad v \equiv \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} = \frac{1}{\sqrt{\mu\epsilon}}.$$

- In that case the second telegrapher's equation demands

$$-\frac{\partial I}{\partial z} = \mathcal{C}\frac{\partial V}{\partial t} = \mathcal{C} f'(t \mp \frac{z}{v})$$

implying an anti-derivative

$$I(z,t) = \pm\mathcal{C}v\, f(t \mp \frac{z}{v}) = \pm\frac{f(t \mp \frac{z}{v})}{Z_o}$$

with

$$Z_o \equiv \frac{1}{\mathcal{C}v} = \frac{\sqrt{\mathcal{L}\mathcal{C}}}{\mathcal{C}} = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \frac{1}{\mathrm{GF}}\sqrt{\frac{\mu}{\epsilon}}.$$

**Telegrapher's equations:**

$$\begin{aligned} -\frac{\partial V}{\partial z} &= \mathcal{L}\frac{\partial I}{\partial t} \\ -\frac{\partial I}{\partial z} &= \mathcal{C}\frac{\partial V}{\partial t} \end{aligned}$$

**where**

$$\mathcal{C} = \epsilon\mathbf{GF}, \ \mathcal{L} = \frac{\mu}{\mathbf{GF}},$$

**with "geometrical factor"**

$$\begin{aligned} \mathbf{GF} &= \frac{W}{d} \ \textbf{parallel-plate} \\ &= \frac{2\pi}{\ln\frac{b}{a}} \ \textbf{coax} \\ &= \frac{\pi}{\cosh^{-1}\frac{D}{2a}} \ \textbf{twin-lead} \end{aligned}$$

- In summary, d'Alembert wave solutions of telegrapher's equations are

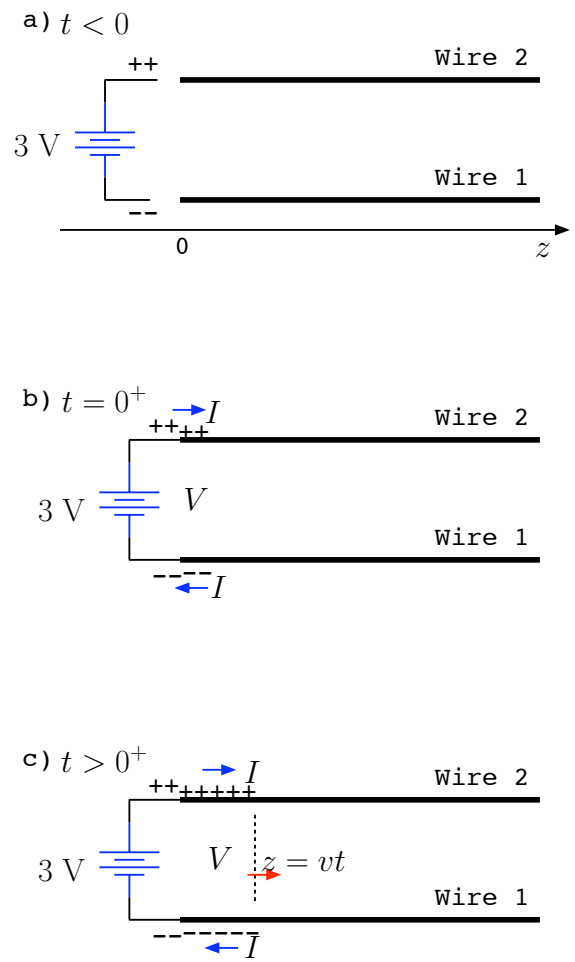
$$V(z,t) = f(t \mp \frac{z}{v}) \quad \text{and} \quad I(z,t) = \pm \frac{f(t \mp \frac{z}{v})}{Z_o}$$

with a propagation speed

$$v = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} = \frac{1}{\sqrt{\mu\epsilon}}$$

that equals the wave speed of the associated electric and magnetic fields, and voltage-to-current ratio representing a **characteristic impedance**

$$Z_o = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \frac{1}{\text{GF}}\sqrt{\frac{\mu}{\epsilon}}.$$




Telegrapher's equations and their d'Alembert solutions provide us with a "handle" on the following **physics**:

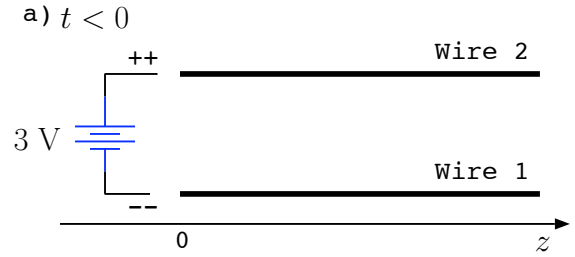
- Suppose that + and - terminals of a 3 V battery makes contact with the terminals of a charge neutral TL at $t = 0$ as depicted in the margin. We will assume that $V(z,t) = I(z,t) = 0$ on the TL for $t < 0$.

As soon as contact is made between the terminals of the battery and the TL, the excess + and - charges on battery terminals will "spill onto" the TL terminals as shown in the figure for $t = 0^+$:

- what really happens is,

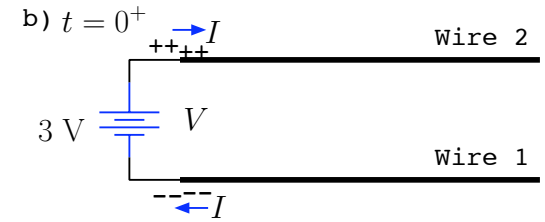
– electrons move from the - terminal of the battery onto the bottom wire of the TL,

– replenished by an equal amount of electrons moving from the top wire into the battery via its + terminal,

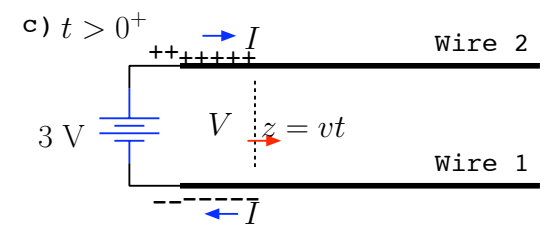



giving the overall impression of current flows $I$ (in opposite direction to electron motion) as marked on the two wires in the diagram.

– currents $I$ and voltage $V$ marked in the diagram are confined only to location $z = 0^+$ at $t = 0^+$, while there is still zero current on the rest of the TL!!!




Having unequal currents on a length of wire is in conflict with our notions from earlier circuit courses, but that's because earlier courses taught us "**lumped-circuit analysis**", an approximate technique justified when it's OK to ignore certain time delays of charge movements in the circuit (when wire lengths are sufficiently short).




Having unequal currents on the TL wire is really what happens

– because, for instance, electrons at some $z > 0$ on the top wire will start moving towards the battery terminal only after the neighboring electrons at $z^-$ deplete the region leaving some excess positive charge.

Thus, currents $I$ on the wires, and voltage $V$ defined and measured across the wires, spread out of $z = 0$ region at a finite speed $v$, in

analogy with ripples spreading out on a pond surface when perturbed by a falling pebble.

- **Telegrapher's equations and their d'Alembert solutions will allow us to calculate how $I$ and $V$ evolve on the TL in quantitative terms.**

To appreciate the distinction between lumped and **distributed circuit analysis**, we next develop a lumped circuit model of a very short length of a TL over which lumped circuit notions may be applicable:

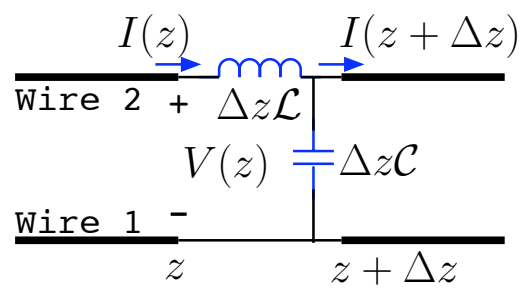
$$-\frac{\partial V}{\partial z} = \mathcal{L}\frac{\partial I}{\partial t}$$
$$-\frac{\partial I}{\partial z} = \mathcal{C}\frac{\partial V}{\partial t}$$

- Let us re-write the first telegrapher's equation as

$$-\Delta V \equiv V(z,t) - V(z+\Delta z,t) = \Delta z \mathcal{L}\frac{\partial I}{\partial t}$$

after approximating the left side as a ratio of infinitesimals.

  - This relation shows that in the current flow direction there is an infinitesimal inductive voltage drop of $\Delta z \mathcal{L}\frac{\partial I}{\partial t}$ between points $z$ and $z+\Delta z$ on the wire carrying current $I \equiv I(z,t) \approx I(z+\Delta z,t)$.




- Likewise, the second equation re-arranged as

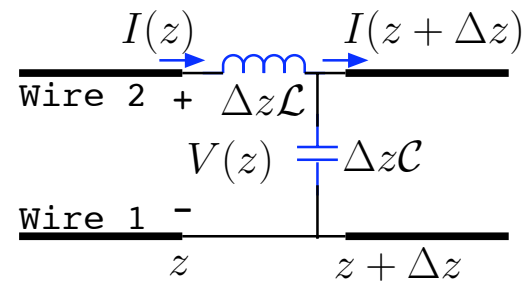
$$-\Delta I \equiv I(z,t) - I(z+\Delta z,t) = \Delta z \mathcal{C}\frac{\partial V}{\partial t},$$

  - this shows that an infinitesimal capacitor current $\Delta z \mathcal{C}\frac{\partial V}{\partial t}$ flows out of a node located between $z$ and $z+\Delta z$ on the wire with current $I$ into a node on the second wire at the same location.

Evidently, a short section $\Delta z$ of the TL has an equivalent T-network with

1. a series inductance $\Delta z\mathcal{L}$ carrying a current $I(z,t) \approx I(z+\Delta z,t)$, and
2. a shunt capacitance $\Delta z\mathcal{C}$ carrying a voltage $V(z+\Delta z,t) \approx V(z,t)$

as shown in the margin.




This lumped-circuit equivalent is only accurate for $\Delta z$ so small that

$$I(z,t) \approx I(z+\Delta z,t) \quad \text{and} \quad V(z+\Delta z,t) \approx V(z,t)$$

are both true, which is of course possible only if $\Delta z \ll \lambda$, $\lambda$ being the shortest wavelength in $I(z,t) \propto \mathbf{H}(z,t)$ and $V(z,t) \propto \mathbf{E}(z,t)$ waveforms.

- Going back to parallel-plate TL in TEM mode, observe that the total power transported in the guide will be the Poynting vector $\mathbf{E} \times \mathbf{H} = E_x H_y \hat{z}$ times the cross-sectional area of the guide, namely, $Wd$.

  Thus, power transported in $z$ direction is

$$\begin{aligned} p(z,t) &= WdE_x(z,t)H_y(z,t), \\ &= (E_x(z,t)d)(H_y(z,t)W) = V(z,t)I(z,t) \end{aligned}$$

  the familiar formula from circuit theory.

Hence, the circuit theory formula

$$P = \frac{1}{2}\mathrm{Re}\{\tilde{V}\tilde{I}^*\}$$

for average power will also hold in sinusoidal-steady state TL problems when we use phasors $\tilde{V}(z)$ and $\tilde{I}(z)$ to represent the $V(z,t)$ and $I(z,t)$ waveforms.

**TL's can also support non-TEM modes having non-zero components of $H_z$ or $E_z$. These modes are non-propagating (evanescent) at low frequencies and remain localized near their excitation regions (e.g., discontinuity points on the line) if $d < \frac{\lambda}{2}$ (pp TL) or if $a+b < \frac{\lambda}{\pi}$ (coax). At high frequencies when these modes cannot be avoided with practical dimensions $d$, $a$, and $b$, it may be practicable to use them rather than the TEM mode. Use single-wire waveguides in that case instead of two-wire TL's.**