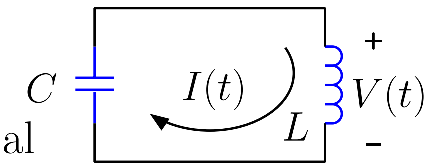
31 Periodic oscillations in lossless TL ckts



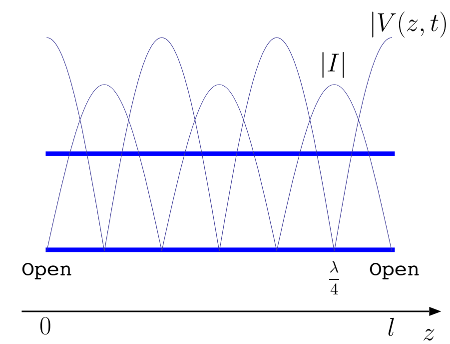
- Lossless LC circuits (see margin) can support source-free and co-sinusoidal voltage and current oscillations at a frequency of

$$\omega = \frac{1}{\sqrt{LC}}$$

known as LC **resonance frequency**.

- Lossless TL circuits can also support source-free voltage and current oscillations, but the number of resonance frequencies is *infinite* and the oscillation waveforms are not restricted to co-sinusoidal forms.
  - Resonance frequencies of lossless TL's are **harmonically related**, and therefore superpositions of resonant oscillations on TL's can add up to arbitrary periodic waveforms as in Fourier series representation of periodic functions.

In this lecture we will examine the periodic oscillations and resonances encountered in lossless and source-free TL circuits.



A TL segment open circuited at both ends can support voltage and current oscillations such that the current waveform vanishes at both ends. Absolute values of a possible set of voltage and current waveforms satisfying this boundary condition are depicted above.

- Consider first a TL segment of some length ℓ having no electrical connection to any elements at either end, as shown in the margin.

  – Effectively, both ends of the TL have been "open circuited", and thus TL current $I(z,t)$ needs to vanish at $z=0$ and $z=\ell$. Since

$$I(z,t)=\frac{f(t-\frac{z}{v})}{Z_o}-\frac{g(t+\frac{z}{v})}{Z_o}$$

  in general, these boundary conditions

$$I(0,t)=\frac{f(t)}{Z_o}-\frac{g(t)}{Z_o}=0$$

  and

$$I(l,t)=\frac{f(t-\frac{\ell}{v})}{Z_o}-\frac{g(t+\frac{\ell}{v})}{Z_o}=0$$

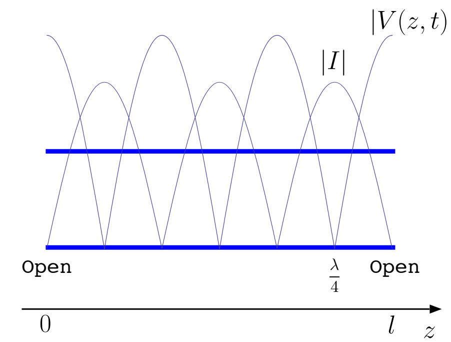
  require that

    ◦ $g(t)=f(t)$
    ◦ $f(t-\frac{\ell}{v})=f(t+\frac{\ell}{v}) \quad\Rightarrow\quad f(t)=f(t+\frac{2\ell}{v})$.

  – the first condition says that forward and backward going waveforms are the same,

  – while the second condition indicates that the waveforms are necessarily periodic with a

    ◦ **period** $T=\frac{2\ell}{v}$
    ◦ ***fundamental*** **frequency** $\omega_o=\frac{2\pi}{T}=\frac{\pi v}{\ell}$.



A TL "stub" open circuited at both ends can support voltage and current oscillations such that the current waveform vanishes at both ends. Absolute values of a possible set of voltage and current waveforms satisfying this boundary condition are depicted above.

Since no other constraint is imposed, any waveform with the specified period is admissible, and the most general such expression is given by the **Fourier series**

$$f(t) = F_o + \sum_{n=1}^{\infty} F_n \cos(n\omega_o t + \theta_n)$$

having harmonically related frequencies $n\omega_o$ and arbitrary **Fourier coefficients** F_n and θ_n.

– Hence, in general, the line current

$$\begin{aligned} I(z,t) &= \frac{f(t-\frac{z}{v}) - f(t+\frac{z}{v})}{Z_o} \\ &= \sum_{n=1}^{\infty} \frac{F_n}{Z_o}[\cos(n\omega_o t + \theta_n - n\beta_o z) - \cos(n\omega_o t + \theta_n + n\beta_o z)] \end{aligned}$$

where $\beta_o \equiv \omega_o/v = \pi/\ell$ is the ***fundamental*** **wavenumber**.

– The same result written in phasor form is

$$\tilde{I}(z) = \sum_{n=1}^{\infty} \frac{F_n}{Z_o} e^{j\theta_n}[e^{-jn\beta_o z} - e^{jn\beta_o z}] = \sum_{n=1}^{\infty} \frac{F_n}{Z_o} e^{j\theta_n}(-2j)\sin(n\beta_o z),$$

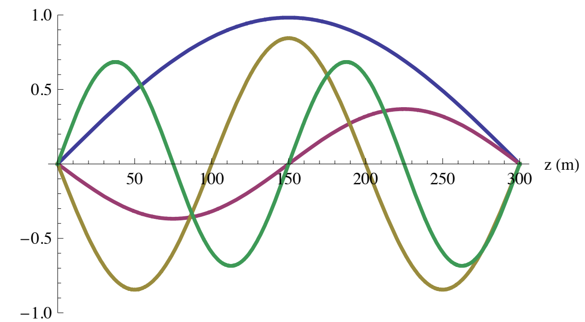
which also means that (back in the time domain)

$$I(z,t) = \sum_{n=1}^{\infty} \frac{2F_n}{Z_o} \sin(n\omega_o t + \theta_n)\sin(n\beta_o z).$$

Also[1]

$$V(z,t) = \sum_{n=1}^{\infty} 2F_n \cos(n\omega_o t + \theta_n)\cos(n\beta_o z)$$

from the phasor $\tilde{V}(z) = \sum_n F_n e^{j\theta_n}[e^{-jn\beta_o z} - e^{jn\beta_o z}]$.



A time-snapshot of the current standing wave modes $n = 1, 2, 3, 4$ on a TL segment 300 m long, open ended on both sides. Each mode n has n half wavelengths fitted into the line length l and high n modes oscillate with higher frequencies. See animation of these modes linked in the class calendar.

In summary:

– Periodic variations of arbitrary complexity — or *timbre*, in analogy with musical instruments — in $V(z,t)$ and $I(z,t)$ are allowed on an open circuited (on both ends) TL segment of length ℓ and consist of superpositions of **resonant modes** (see margin)

$$\cos(n\frac{\pi v}{\ell}t + \theta_n)\cos(n\frac{\pi}{\ell}z) \quad \text{and} \quad \sin(n\frac{\pi v}{\ell}t + \theta_n)\sin(n\frac{\pi}{\ell}z),$$

respectively, in the range $n \geq 1$, each one being a **standing wave**.

– Each resonant mode or standing wave of index $n \geq 1$ has a

∘ resonance frequency

$$\omega = \frac{\pi v}{\ell} n \text{ rad/s} \quad \text{or} \quad f = \frac{v}{2\ell} n \text{ Hz}$$

∘ resonance wavelength

$$\lambda = \frac{v}{f} = \frac{2\ell}{n},$$

[1] Note that an arbitrary DC term can also be included in $V(z,t)$.

implying that

$$\ell = n\frac{\lambda}{2},$$

that is, **the line length is an integer multiple of half-wavelength at each resonance**.

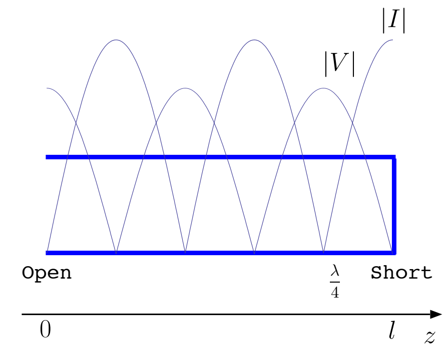
- The resonances examined above also apply to TL's of length ℓ **shorted** at both ends, provided that the mode equations above are swapped between voltage and current — that is, periodic variations of arbitrary complexity in $I(z,t)$ and $V(z,t)$ consist of superpositions of **resonant modes**

  $$\cos(n\frac{\pi v}{\ell}t + \theta_n)\cos(n\frac{\pi}{\ell}z) \quad \text{and} \quad \sin(n\frac{\pi v}{\ell}t + \theta_n)\sin(n\frac{\pi}{\ell}z),$$

  respectively, in the range $n \geq 1$.

  Note that in this case the voltage modes vanish at $z = 0$ and $z = \ell$ as required by the boundary condition $V(0,t) = V(\ell,t) = 0$ imposed by having shorts at both ends.

- For TL's of length ℓ **open at one end shorted at the other** end, resonant wavelengths and frequencies can be identified by requiring ℓ to be an odd multiple of $\frac{\lambda}{4}$
  - the reason for this is, the nulls of waveforms $\propto \cos(\beta z)$ and $\sin(\beta z)$ are separated by odd multiples of
$$\frac{\lambda}{4} = \frac{2\pi/\beta}{4} = \frac{\pi}{2\beta}.$$
  - Hence, resonance condition is
$$\ell = \frac{\lambda}{4}(2n+1), \quad n \geq 0,$$
and since
$$f\lambda = v$$
it follows that the resonance frequencies are
$$f = \frac{v}{2\ell}(\frac{1}{2}+n) \quad \text{and} \quad \omega = \frac{\pi v}{\ell}(\frac{1}{2}+n) \text{ for } n \geq 0.$$



A TL stub open at one end short at the other can support voltage and current oscillations such that the current waveform vanishes at the open end while the voltage waveform vanishes at the shorted end. Absolute values of a possible set of voltage and current waveforms satisfying this boundary condition are depicted above. Resonant standing waves modes on this line will have voltage and current nulls separated by an odd multiple of a quarter wavelength.

Example 1: A lossless TL of 600 m length is left open at $z = 0$ and shorted at $z = l = 600$ m. Determine (a) resonant frequencies of the line, (b) resonant voltage modes, (c) resonant current modes obtained from the voltage modes using the telegrapher's equations. The line has a characteristic impedance of $Z_o = 50\,\Omega$ and a propagation velocity $v = c$.

Solution: (a) The line must be an odd multiple of quarter wavelengths at the resonant frequencies. Therefore,

$$600\,\mathrm{m} = (2n+1)\frac{\lambda}{4} \quad \Rightarrow \quad 600\,\mathrm{m} = (2n+1)\frac{c/f}{4}$$

leading to

$$f = (2n+1)\frac{300\,\mathrm{m}/\mu\mathrm{s}}{4 \cdot 600\,\mathrm{m}} = (2n+1)\frac{1}{8}\,\mathrm{MHz}, \;\; n \geq 0.$$

(b) Since the current modes need to vanish at $z = 0$, we can express them in terms of a sine function as

$$\sin(\beta z)\sin(\omega t)$$

where

$$\omega = 2\pi f = (2n+1)\frac{\pi}{4}\,\mathrm{Mrad/s},$$

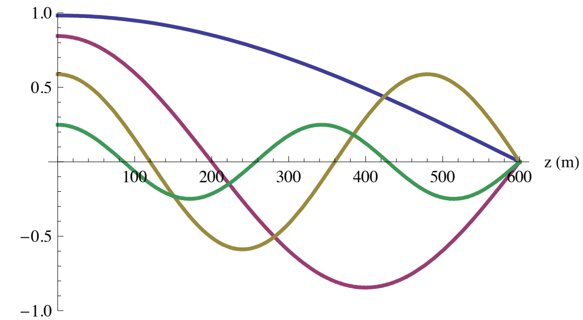
and

$$\beta = \frac{2\pi}{\lambda} = (2n+1)\frac{\pi}{1200}\,\mathrm{rad/m}.$$

In explicit terms, current modes are

$$I_n(z,t) = \sin((2n+1)\frac{\pi}{1200}z)\sin((2n+1)\frac{\pi}{4}t)$$

where z is in m and t in μs.



A time-snapshot of the voltage standing wave modes $n = 0, 1, 2, 3$ on a TL segment 600 m long, open ended at $z = 0$ and shorted at $z = 600$ m. Each mode n has $2n+1$ quarter wavelengths fitted into the line length l and the high n modes oscillate with higher frequencies. See animation of these modes linked in the class calendar.

(c) Let's find the voltage modes $V_n(z,t)$ from the current modes, above using one of the telegrapher's equations,

$$-\frac{\partial V}{\partial z} = \mathcal{L}\frac{\partial I}{\partial t}.$$

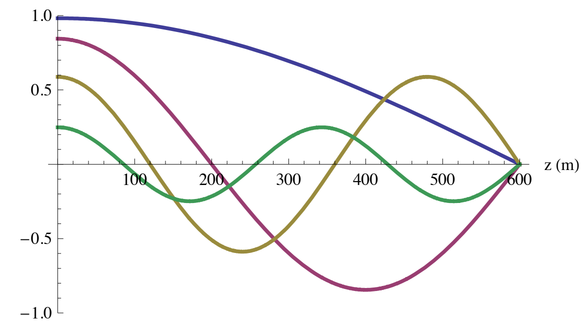
Substituting $I_n(z,t)$ into this equation and differentiating we find

$$\frac{\partial V}{\partial z} = -(2n+1)\frac{\pi}{4}\mathcal{L}\sin((2n+1)\frac{\pi}{1200}z)\cos((2n+1)\frac{\pi}{4}t).$$

Next finding the anti-derivative of the above, we conclude

$$V_n(z,t) = 300\mathcal{L}\cos((2n+1)\frac{\pi}{1200}z)\cos((2n+1)\frac{\pi}{4}t).$$

A snapshot of the animation of resonant voltage modes is shown in the margin.



- **How can one get source free oscillations in a TL?**

  One answer is, the TL might have been connected to a source in the past before being disconnected from it.

  - Consider the circuit shown in the margin where a 3V battery is switched in and out for 1 μs on a line of length $l = 600$ m.

  For $v = c$, we can write the voltage and the current on the line at $t = 1\,\mu$s (by inspection) as

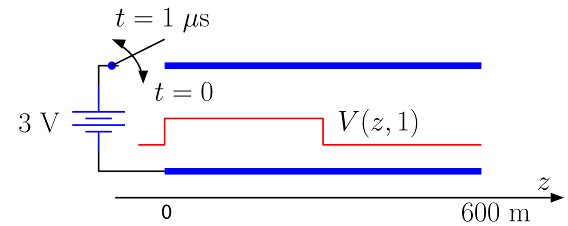
$$V(z,1) = 3\text{rect}(\frac{z-150}{300})\text{ V} \quad \text{and} \quad I(z,1) = \frac{3\text{rect}(\frac{z-150}{300})}{Z_o}\text{ A}.$$

  After $t = 1\,\mu$s both ends of the TL will be open, and, therefore, only periodic waveforms with a

  - fundamental period of $T = \frac{2\ell}{v} = \frac{1200}{300} = 4\,\mu s$ and
  - fundamental frequency of $\omega_o = \frac{2\pi}{T} = \frac{\pi}{2}$ Mrad/s

  will be allowed on the source free line.

  Therefore, $V(z,t)$ and $I(z,t)$ for $t > 1\,\mu$s can be expressed as a weighted superposition of the resonant modes of the line with resonant frequencies $n\omega_o$, subject to the initial conditions $V(z,1)$ and $I(z,1)$ given above.



Also, resistors R at temperature T connected to TL terminals can transfer thermal noise energy to the TL. If the resistors are disconnected at some point in time, the energy left on the TL will be shared between its resonant modes (up to a frequency limit $KT/\hbar$ imposed by quantum mechanics) at an average level of KT joules (per mode) where K is the *Boltzmann constant*. Lossy lines with finite conductivity also produce thermal noise. Thermal noise is easy to detect and routinely interferes with weak communication signals that we care about!