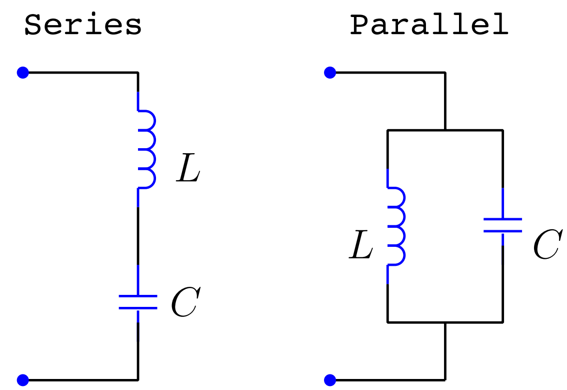
32 Input impedance and microwave resonators



- The **input impedance** and **admittance** of the **series** and **parallel** LC resonators shown in the margin are, respectively,

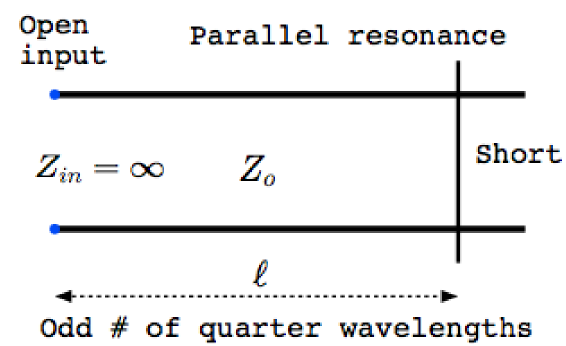
$$Z_s = j(\omega L - \frac{1}{\omega C}) \quad \text{and} \quad Y_p = j(\omega C - \frac{1}{\omega L}),$$

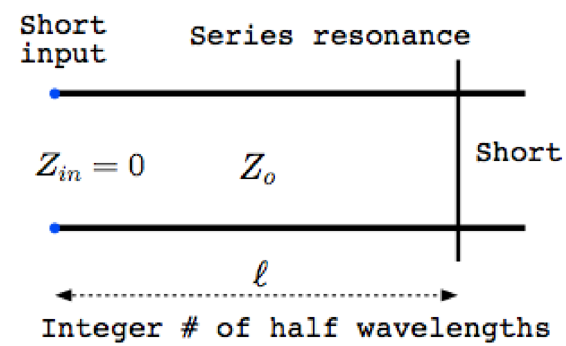
  both of which vanish at the common resonance frequency of these networks, namely

$$\omega = \frac{1}{\sqrt{LC}} \equiv \omega_o.$$

  – Recall that LC resonators play an important role in the design of filter and tuning circuits.

In this lecture we will examine the input impedance of microwave resonators consisting of open or short circuited TL stubs.



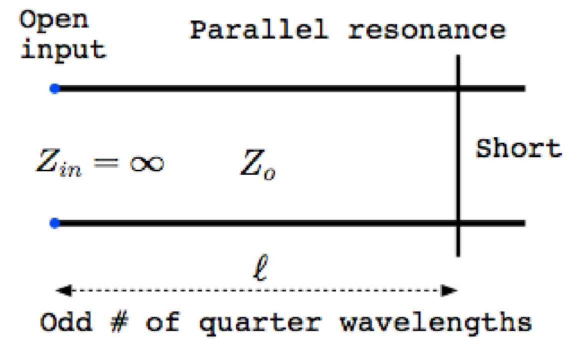


- In the last lecture we learned that when a shorted stub is open circuited at its input port, it shows resonance if the stub length ℓ is an odd multiple of $\frac{\lambda}{4}$.

  – The corresponding resonant frequencies are

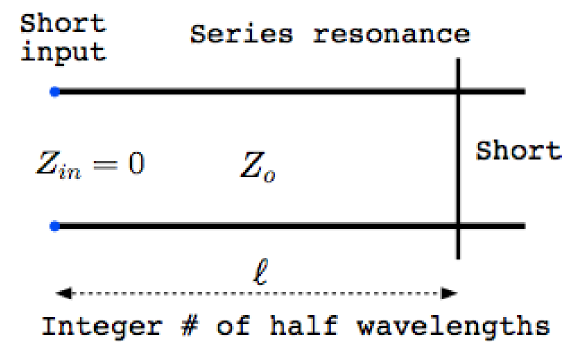
$$f = \frac{v}{2\ell}(\frac{1}{2} + n) \ \text{ for } \ n = 0, 1, 2, 3, \cdots$$

and the input port of the stub coincides with a voltage max and a current null, i.e., $I(0,t) = 0$ — thus the input impedance Z_{in} of the stub is *infinite* at these resonances, just like the impedance of the *parallel* LC-circuit depicted above.



  – Thus this set of resonant frequencies are referred to as **parallel resonances** of the shorted stub.

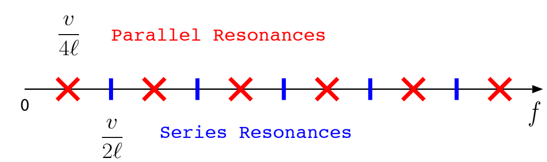
- We also learned that when the stub length ℓ is an an integer multiple of $\frac{\lambda}{2}$, its voltage at the input terminal is necessarily zero, implying that the input impedance Z_{in} must also be *zero*.



  – The corresponding resonant frequencies are

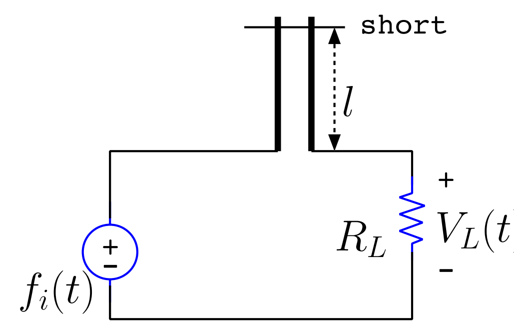
$$f = \frac{v}{2\ell} n \ \text{ for } \ n = 1, 2, 3, \cdots$$

  and are termed **series resonances** of the shorted stub, in analogy with the zero impedance of the *series* LC-circuit depicted above.



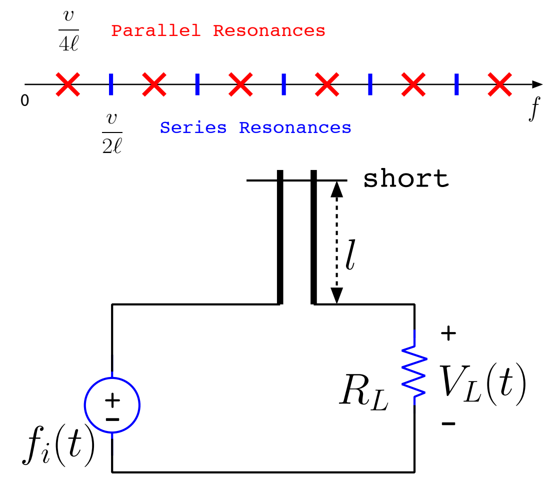
The diagram in the margin marks the locations of *parallel* and *series* resonance frequencies of the shorted stub associated with *infinite* and *zero* input impedance Z_{in}.

Thus, a shorted stub, included in a circuit such as the one shown in the margin, will exhibit extreme behavior at these special frequencies — namely it will appear as a



- short at its series resonances, causing the entire input signal $f_i(t)$ to appear as $V_L(t)$ across the load R_L, and

- open at its parallel resonances, causing $V_L(t)$ across the load R_L to be notched out.



We next focus our attention on how Z_{in} of the stub appears at other frequencies not coinciding with any of the resonances discussed above.

- In the following we will assume that the TL stub, as well as the circuit it is connected to, are all in sinusoidal steady state at a frequency determined by the frequency of the sinusoidal source $f_i(t)$.

  – In that case d'Alembert solutions will also be co-sinusoidal at the source frequency ω and we can express $V(z,t)$ and $I(z,t)$ on the line as

$$V(z,t) = \mathrm{Re}\{V^+ e^{j\omega(t-\frac{z}{v})}\} + \mathrm{Re}\{V^- e^{j\omega(t+\frac{z}{v})}\} \quad \Leftrightarrow \quad \tilde{V}(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}$$

and

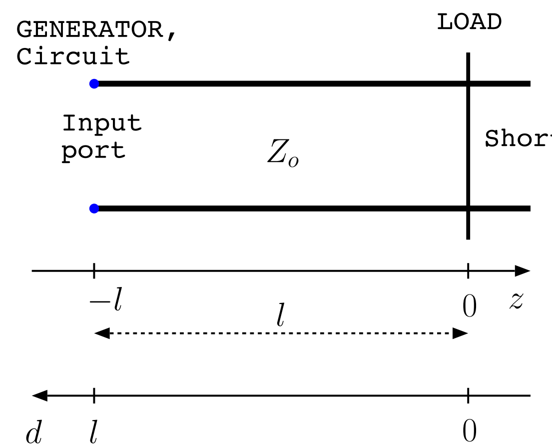
$$I(z,t) = \frac{\mathrm{Re}\{V^+ e^{j\omega(t-\frac{z}{v})}\} - \mathrm{Re}\{V^- e^{j\omega(t+\frac{z}{v})}\}}{Z_o} \quad \Leftrightarrow \quad \tilde{I}(z) = \frac{V^+ e^{-j\beta z} - V^- e^{j\beta z}}{Z_o},$$

where

  – $\beta = \frac{\omega}{v} = \omega\sqrt{\mathcal{LC}}$ is the wavenumber at frequency ω, and

– V^+ and V^- are phasors of forward and backward propagating voltage waves on the line evaluated at $z = 0$.

We have expressed the phasor counterparts of co-sinusoidal waves $V(z,t)$ and $I(z,t)$ above on the right, for it will be necessary to use phasors in defining an input impedance — the impedance concept belongs to the frequency domain!



- Before applying the boundary condition at the shorted end of the TL stub, it will be convenient to shift the origin of our coordinate system to coincide with the shorted termination rather than the input port of the TL.

- It will also be convenient to refer to "$-z$" as "d", with the variable d growing to the left from the short termination toward the input terminal of the line.

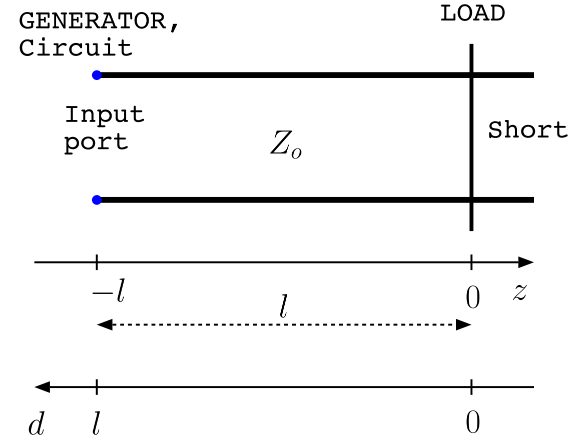
  In that case, the input impedance of the shorted stub can be denoted as

$$Z(l) = \frac{\tilde{V}(d=l)}{\tilde{I}(d=l)},$$

where

$$\tilde{V}(d) \equiv V^+ e^{j\beta d} + V^- e^{-j\beta d} \quad \text{and} \quad \tilde{I}(d) \equiv \frac{V^+ e^{j\beta d} - V^- e^{-j\beta d}}{Z_o}.$$

- An immediate benefit of our new notation comes when we apply the voltage boundary condition at the short termination.



– We apply it as

$$V(0,t)=0 \quad \Leftrightarrow \quad \tilde{V}(0)=V^{+}+V^{-}=0$$

from which it follows that

$$V^{-}=-V^{+}$$

and thus

$$\tilde{V}(d) \equiv V^{+}(e^{j\beta d}-e^{-j\beta d})=j2V^{+}\sin(\beta d)$$

and

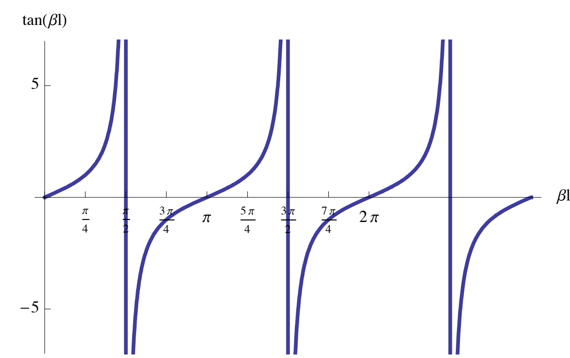
$$\tilde{I}(d) \equiv \frac{V^{+}(e^{j\beta d}+e^{-j\beta d})}{Z_o}=Y_o 2V^{+}\cos(\beta d),$$

where

$$Y_o \equiv \frac{1}{Z_o} \text{ Characteristic admittance.}$$

– Finally the **input impedance** of the shorted stub is

$$Z(l)=\frac{\tilde{V}(l)}{\tilde{I}(l)}=jZ_o\tan(\beta l).$$



Note that: input impedance $Z(l)=0+jX(l)$ is (see margin for $X(l)$)

1. purely reactive for all l,
2. has a positive imaginary part and therefore it is **inductive** for

$$\beta l=\frac{2\pi}{\lambda}l<\frac{\pi}{2}\text{ rad}=90^{o} \quad \Rightarrow \quad 0<l<\frac{\lambda}{4}=\text{Quarter wavelength.}$$

3. has a negative imaginary part and therefore it is **capacitive** for

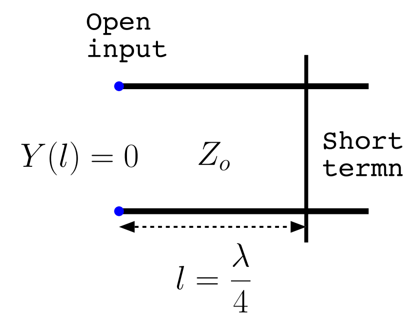
$$\frac{\pi}{2} < \beta l = \frac{2\pi}{\lambda} l < \pi \text{ rad} = 180^o \quad \Rightarrow \quad \frac{\lambda}{4} < l < \frac{\lambda}{2} = \text{Half wavelength.}$$

4. is periodic with a period of $\frac{\lambda}{2}$ over l, which means that

   all possible **reactive impedances** of the form jX are realized for $0 < l < \frac{\lambda}{2}$.

   - a shorted TL stub of length $0 < l < \frac{\lambda}{2}$ spans all possible impedances that can be provided by all possible inductors and capacitors!

   for a length $l \ll \frac{\lambda}{4}$ shorted stub is a pure inductor with impedance

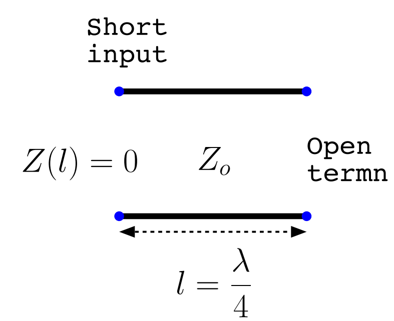


$$Z(l) = jZ_o \tan(\beta l) \approx jZ_o \beta l = j\sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\omega\sqrt{\mathcal{L}\mathcal{C}}l = j\omega\mathcal{L}l.$$

   - here we used $\tan(\beta l) \approx \beta l$, which is valid when $\beta l \ll 1$ in radians.

5. at $l = \frac{\lambda}{4}$ the **input admittance** of the shorted stub,

$$Y(l) = \frac{1}{Z(l)} = \frac{1}{jZ_o \tan(\beta l)} = -jY_o \cot(\beta l),$$

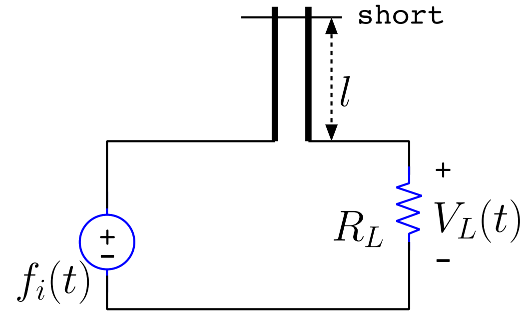


   vanishes, meaning that

   - a shorted stub of length $l = \frac{\lambda}{4}$ appears at its input terminals like an *open* (see margin).

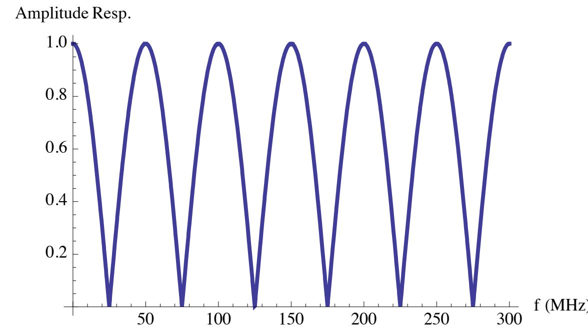
6. at $l = \frac{\lambda}{2}$ the **input impedance** of the shorted stub returns back to zero, which in turn indicates, in view of (5), that

    - an open ended stub of length $l = \frac{\lambda}{4}$ must appear at its input terminals like a *short* (see margin).



Next set of examples illustrate the uses of shorted/opened TL stubs as circuit elements.

Example 1: A shorted TL stub of length $l = 3$ m is connected in *series* with a resistor $R_L = 50\,\Omega$ as shown in the diagram in the margin. Plot the magnitude of the frequency response $H(\omega) = \frac{\tilde{V}_L}{\tilde{F}}$ as a function of frequency $f = \frac{\omega}{2\pi}$ if $Z_o = 50\,\Omega$ and $v = c$ on the stub. Interpret the amplitude response curve $|H(\omega)|$ in terms of resonance frequencies of the shorted line.



Solution: Using $\beta = \frac{\omega}{c}$ and voltage division, we find that frequency response

$$H(\omega) = \frac{\tilde{V}_L}{\tilde{F}_i} = \frac{R_L}{R_L + jZ_o \tan(\beta l)} = \frac{1}{1 + j\tan(\frac{\omega}{c} l)}.$$

The plot of $|H(\omega)|$ with the given parameters is shown in the margin. The peaks of the amplitude response occur at the series resonance frequencies of the shorted stub when its input impedance is zero (an effective short). The nulls of the amplitude response correspond to parallel resonances of the stub when it appears like an open at its input terminals.

Example 2: Consider a shorted TL connected at $d = l$ to an inductor L. Determine the resonances of the combined network.

Solution: The input impedance of the shorted line is

$$Z(l) = jZ_o \tan(\beta l) = jZ_o \tan(\omega\sqrt{\mathcal{LC}}l)$$

whereas inductor L has an impedance $Z_L = j\omega L$. If the inductor and shorted stub are connected in series (see margin), then the *series resonances* of the network will be observed when the network input impedance

$$Z_L + Z(l) = j\omega L + jZ_o \tan(\omega\sqrt{\mathcal{LC}}l)$$

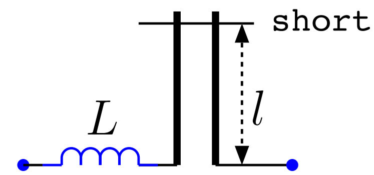
equals zero. The *parallel resonances* of the network will be observed when the impedance is infinite. While the series resonance frequencies of the stub will be shifted because of the inductor, parallel resonances will not shift (infinities due to tan function cannot be shifted by the finite additive term due to the inductor). The shifted series resonance frequencies ω_n can be found graphically by plotting $Z_L + Z(l)$ and looking for the zero crossings.

If the inductor and shorted stub are connected in parallel, then the *parallel resonances* of the network will be observed when the network input admittance

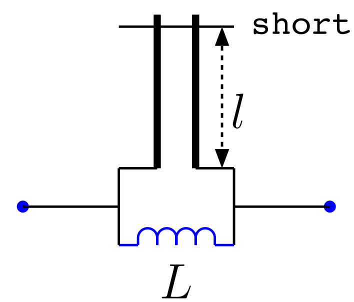
$$Y_L + Y(l) = \frac{1}{j\omega L} + \frac{1}{jZ_o \tan(\omega\sqrt{\mathcal{LC}}l)}$$

equals zero (same as infinite input impedance). The *series resonances*, on the other hand, will be observed when the admittance is infinite (same as zero input impedance). Series resonances of the stub will not be shifted with, unlike its parallel resonances. The shifted parallel resonance frequencies ω_n will equal the series resonance frequencies of the series connected network described above.

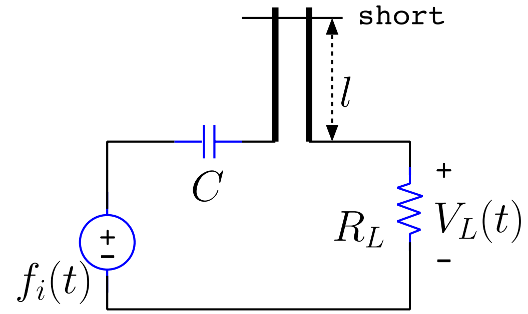
`Series network:`



`Parallel network:`



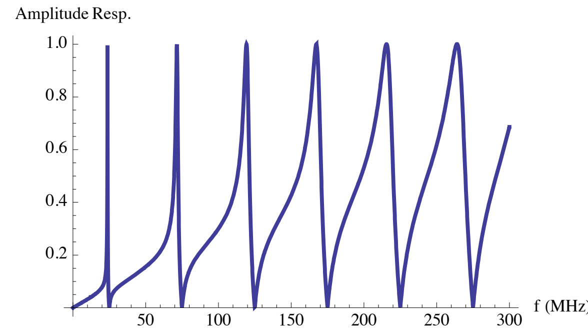
Example 3: A shorted TL stub of length $l = 3$ m is connected in *series* with a a capacitor $C = 10$ pf and a resistor $R_L = 50\,\Omega$ as shown in the diagram in the margin. Plot the magnitude of the frequency response $H(\omega) = \frac{\tilde{V}_L}{\tilde{F}}$ as a function of frequency $f = \frac{\omega}{2\pi}$ if $Z_o = 50\,\Omega$ and $v = c$ on the stub. Interpret the amplitude response curve $|H(\omega)|$ in terms of resonance frequencies of the shorted line.



Solution: Using $\beta = \frac{\omega}{c}$ and voltage division, we find that frequency response

$$H(\omega) = \frac{\tilde{V}_L}{\tilde{F}_i} = \frac{R_L}{R_L + \frac{1}{j\omega C} + jZ_o\tan(\beta l)} = \frac{1}{1 + \frac{1}{j\omega R_L C} + j\tan(\frac{\omega}{c}l)}.$$

The plot of $|H(\omega)|$ with the given parameters is shown in the margin. The peaks of the amplitude response occur at the *shifted* series resonance frequencies of the shorted T.L. stub. The nulls of the amplitude response correspond to parallel resonances of the stub when it appears open.



Example 4:

(a) If in the TL circuit shown in the margin $I_R = 2\angle 0^o$ A, what is the line length l in terms of wavelength λ of the given source frequency on the line?

(b) Repeat for $I_R = 0$.

Note: starting in this example we are dropping the tildes on the phasors.

Solution:

(a) If $I_R = 2\angle 0^o$ A, then KCL application at the source terminal implies that $I(l) = 0$.

In that case the TL has an *open* at $d = l$. Since $d = 0$ is also an open, we need to have l to be an integer multiple of $\frac{\lambda}{2}$.

In other words, the condition of $I_R = 2\angle 0^o$ will only be realized in the above circuit if the source frequency is such that the TL length l happens to be some integer multiple of $\frac{\lambda}{2}$ at the given frequency.

(b) If $I_R = 0$, then $V(l) = (50\,\Omega)I_R = 0$, implying that the T.L. has a *short* at $d = l$.

Since $d = 0$ is an open, we need to have l some odd multiple of $\frac{\lambda}{4}$.