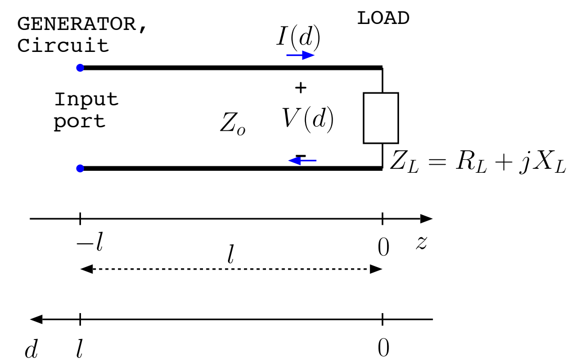
33 TL circuits with half- and quarter-wave transformers



- Last lecture we established that phasor solutions of telegrapher's equations for TL's in sinusoidal steady-state can be expressed as

$$V(d) = V^+ e^{j\beta d} + V^- e^{-j\beta d} \quad \text{and} \quad I(d) = \frac{V^+ e^{j\beta d} - V^- e^{-j\beta d}}{Z_o}$$

  in a new coordinate system shown in the margin.

  By convention the load is located on the right at $z = 0 = d$, and the TL input connected to a generator or some source circuit is shown on the left at $d = l$.

  We have replaced the short termination of the previous lecture with an arbitrary load impedance

$$Z_L = R_L + jX_L.$$

  In this lecture we will discuss sinusoidal steady-state TL circuit problems having arbitrary reactive loads but with line lengths l constrained to be integer multiples of $\frac{\lambda}{4}$ (at the operation frequency).

  The constraint will be lifted next lecture when we will develop the general analysis tools for sinusoidal steady-state TL circuits.

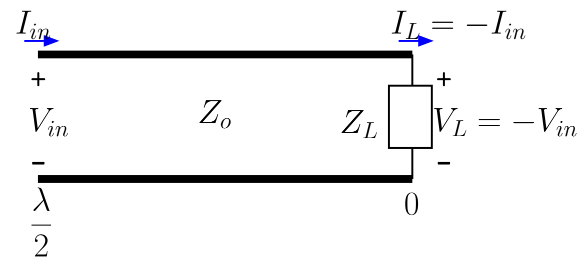
- In the TL circuit shown in the margin an arbitrary load Z_L is connected to a TL of length $l = \frac{\lambda}{2}$ at the source frequency.

  Given that
  $$e^{\pm j\beta\frac{\lambda}{2}} = e^{\pm j\frac{2\pi}{\lambda}\frac{\lambda}{2}} = e^{\pm j\pi} = -1,$$
  the general phasor relations
  $$V(d) = V^+e^{j\beta d} + V^-e^{-j\beta d} \text{ and } I(d) = \frac{V^+e^{j\beta d} - V^-e^{-j\beta d}}{Z_o}$$
  imply
  $$V_{in} \equiv V(\frac{\lambda}{2}) = -V^+ - V^- = -V(0) = -V_L,$$
  $$I_{in} \equiv I(\frac{\lambda}{2}) = \frac{-V^+ + V^-}{Z_o} = -I(0) = -I_L.$$
  We conclude that a $\frac{\lambda}{2}$**-transformer**

  – inverts the algebraic sign of its voltage and current inputs at the load end (and vice versa), and

  – has an input impedance identical with the load impedance since
  $$Z_{in} \equiv \frac{V_{in}}{I_{in}} = \frac{-V_L}{-I_L} = Z_L.$$
  These very simple results are easy to remember and use.

Half-wave transformer:

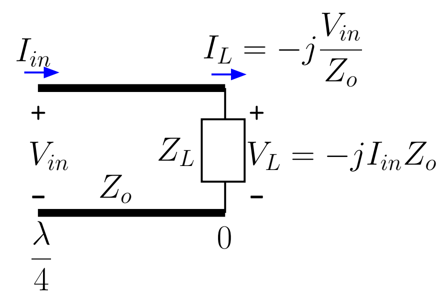


- In the TL circuit shown in the margin an arbitrary load Z_L is connected to a TL of length $l = \frac{\lambda}{4}$ at the source frequency.

  Given that
  $$e^{\pm j\beta\frac{\lambda}{4}} = e^{\pm j\frac{2\pi}{\lambda}\frac{\lambda}{4}} = e^{\pm j\frac{\pi}{2}} = \pm j,$$
  general phasor relations
  $$V(d) = V^+e^{j\beta d} + V^-e^{-j\beta d} \text{ and } I(d) = \frac{V^+e^{j\beta d} - V^-e^{-j\beta d}}{Z_o}$$
  imply
  $$V_{in} \equiv V(\frac{\lambda}{4}) = jV^+ - jV^- = jI(0)Z_o = jI_LZ_o,$$
  $$I_{in} \equiv I(\frac{\lambda}{4}) = \frac{jV^+ + jV^-}{Z_o} = j\frac{V(0)}{Z_o} = j\frac{V_L}{Z_o}.$$
  We conclude that a $\frac{\lambda}{4}$**-transformer**

  - has an input impedance
  $$Z_{in} \equiv \frac{V_{in}}{I_{in}} = \frac{jI_LZ_o}{jV_L/Z_o} = \frac{Z_o^2}{V_L/I_L} = \frac{Z_o^2}{Z_L},$$
  - and provides a load current
  $$I_L = -j\frac{V_{in}}{Z_o},$$
  proportional to input voltage V_{in} but independent of load impedance Z_L.

Quarter-wave transformer:



$Z_{in}Z_L = Z_o^2$

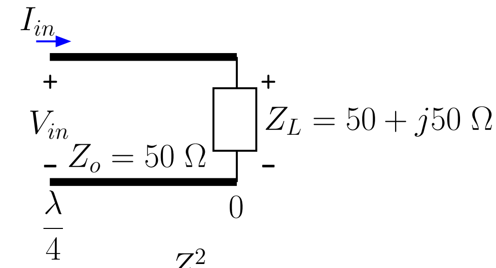
Quarter-wave current-forcing equation:

$$I_L = -j\frac{V_{in}}{Z_o}.$$

Load voltage

$$V_L = Z_LI_L$$

once I_L is available from above equation.



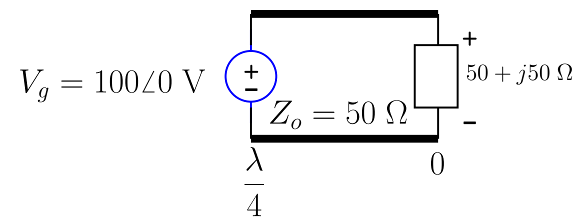
$$Z_{in} = \frac{Z_o^2}{Z_L}$$

Example 1: Given $Z_L = 50 + j50\,\Omega$, what is Z_{in} for a $\frac{\lambda}{4}$ transformer with $Z_o = 50\,\Omega$?

Solution: It is

$$Z_{in} = \frac{Z_o^2}{Z_L} = \frac{50^2}{50 + j50} = \frac{50}{1 + j1} = \frac{50}{1 + j1}\frac{1 - j1}{1 - j1} = 25 - j25\,\Omega.$$

Notice that an *inductive* Z_L has been turned into a *capacitive* Z_{in} by $\frac{\lambda}{4}$ transformer.



Example 2: The load and the transformer of Example 1 are connected to a source with voltage phasor $V_g = 100\angle 0^o$ V at the input port. What is the load current I_L and what is the average power absorbed by the load?

Solution: Since $V_{in} = V_g = 100\angle 0^o\,$V, the current-forcing formula for the quarter-wave transformer implies

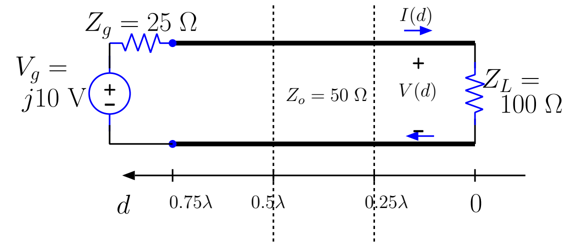
$$I_L = -j\frac{V_{in}}{Z_o} = -j\frac{100}{50} = -j2\,\text{A}.$$

To find the average power absorbed, we first note that load voltage

$$V_L = Z_L I_L = (50 + j50)(-j2) = 100 - j100\,\text{V}.$$

Thus,

$$P_L = \frac{1}{2}\text{Re}\{V_L I_L^*\} = \frac{1}{2}\text{Re}\{(100 - j100)(j2)\} = 100\,\text{W}.$$



Example 3: Load $Z_L = 100\,\Omega$ is connected to a T.L. with length $l = 0.75\lambda$. At the generator end, $d = 0.75\lambda$, a source with open circuit voltage $V_g = j10$ V and Thevenin impedance $Z_g = 25\,\Omega$ is connected. Determine V_L and I_L if $Z_o = 50\,\Omega$.

Solution: First we determine input impedance Z_{in} by noting that $Z_L = 100\,\Omega$ transforms to itself, namely $100\,\Omega$ at $d = 0.5\lambda$, but then it transforms from $d = 0.5\lambda$ to 0.75λ as

$$Z_{in} = \frac{Z_o^2}{Z(0.5\lambda)} = \frac{50^2}{100} = 25\,\Omega.$$

Hence, using voltage division, we find,

$$V_{in} = V_g \frac{Z_{in}}{Z_g + Z_{in}} = j10 \frac{25}{25+25} = j5\,\text{V}.$$

Next, using half-wave transformer rule, we notice that

$$V(0.25\lambda) = -V_{in} = -j5\,\text{V},$$

and finally applying the quarter-wave current forcing equation with $V(0.25\lambda)$ we get

$$I_L = -j\frac{V(0.25\lambda)}{Z_o} = -j\frac{-j5}{50} = -0.1\,\text{A}.$$

Clearly, then, the load voltage is

$$V_L = Z_L I_L = (100\,\Omega)(-0.1\,\text{A}) = -10\,\text{V}.$$

Example 4: In the circuit shown in the margin, $Z_{L1} = 50\,\Omega$, $Z_{L2} = 100\,\Omega$, and $Z_{o1} = Z_{o2} = 50\,\Omega$. Determine I_{L1} and I_{L2} if $V_{in} = 5\,\text{V}$. Both T.L. sections are quarter-wave transformers.

Solution: Using the current-forcing equation, we have

$$I_{L1} = I_{L2} = -j\frac{V_{in}}{Z_o} = -j\frac{5}{50} = -j0.1\,\text{A}.$$

Consequently,

$$V_{L1} = I_{L1}Z_{L1} = -j0.1\,\text{A} \times 50\,\Omega = -j5\,\text{V}$$

and

$$V_{L2} = I_{L2}Z_{L2} = -j0.1\,\text{A} \times 100\,\Omega = -j10\,\text{V}.$$

Thus, total avg power absorbed is

$$\begin{aligned} P &= \frac{1}{2}\text{Re}\{V_{L1}I_{L1}^*\} + \frac{1}{2}\text{Re}\{V_{L2}I_{L2}^*\} \\ &= = \frac{1}{2}\text{Re}\{-j5 \times j0.1\} + \frac{1}{2}\text{Re}\{-j10 \times j0.1\} = 0.75\,\text{W}. \end{aligned}$$