4 Time harmonic sources and *retarded* potentials

- The solution of forced wave equation

$$\nabla^2 V - \mu_o \epsilon_o \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_o}$$

  for scalar potential V is most conveniently obtained in the frequency domain:

  Consider a time-harmonic forcing function ρ and a time-harmonic response V expressed as

$$\rho(\mathbf{r}, t) = \text{Re}\{\tilde{\rho}(\mathbf{r}) e^{j\omega t}\} \text{ and } V(\mathbf{r}, t) = \text{Re}\{\tilde{V}(\mathbf{r}) e^{j\omega t}\}$$

  in terms of phasors

$$\tilde{\rho}(\mathbf{r}) \text{ and } \tilde{V}(\mathbf{r}).$$

  Then, the above wave equation transforms — upon replacing $\frac{\partial}{\partial t}$ by $j\omega$ — into phasor form as

$$\nabla^2 \tilde{V} + \mu_o \epsilon_o \omega^2 \tilde{V} = -\frac{\tilde{\rho}}{\epsilon_o}.$$

- For $\omega = 0$ the above equation reduces to Poisson's equation, which we know has, with an impulse forcing

$$\tilde{\rho}(\mathbf{r}) = \delta(\mathbf{r}), \text{ an impulse response solution } \tilde{V}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o |\mathbf{r}|} \equiv \frac{1}{4\pi\epsilon_o r}$$

where $r \equiv |\mathbf{r}|$ denotes the distance of the observing point $\mathbf{r}$ from the impulse point located at the origin.

– Note that this impulse response $\propto 1/r$ is symmetric with respect to the origin just like the impulse input $\delta(\mathbf{r})$.

Note that by substituting the source function $\delta(\mathbf{r})$ and response function $\frac{1}{4\pi\epsilon_o r}$ back into the Poisson's equation we obtain an equality

$$\nabla^2\left(\frac{1}{|\mathbf{r}|}\right) = -4\pi\delta(\mathbf{r}),$$

which is a useful vector identity.

We now *postulate* and subsequently prove that for $\omega \geq 0$, the impulse response solution of the forced wave equation — i.e., with forcing function $\tilde{\rho}(\mathbf{r}) = \delta(\mathbf{r})$ — is

$$\tilde{V}(\mathbf{r}) = \frac{e^{-jk|\mathbf{r}|}}{4\pi\epsilon_o|\mathbf{r}|} \quad \text{with} \quad k \equiv \omega\sqrt{\mu_o\epsilon_o} = \frac{\omega}{c}.$$

Proof: For $\tilde{\rho}(\mathbf{r}) = \delta(\mathbf{r})$ the source of the forced wave equation (for an arbitrary ω) is *symmetric with respect to the origin*, implying that the corresponding solution $\tilde{V}(\mathbf{r})$ should also have the same type of symmetry. Then, with *no loss of generality*, we can claim a solution for the case $\tilde{\rho}(\mathbf{r}) = \delta(\mathbf{r})$ of the form

$$\tilde{V}(\mathbf{r}) = \frac{f(r)}{r}$$

where

- $f(r) = \frac{1}{4\pi\epsilon_o}$ for $\omega = 0$, and
- $f(r)$ is *to be determined* for an arbitrary ω as follows:

– Substituting $f(r)/r$ for $\tilde{V}(r)$ and $\delta(\mathbf{r})$ for $\rho(\mathbf{r})$ in the forced wave equation (see margin), we obtain

$$\nabla^2(\frac{f(r)}{r}) + k^2\frac{f(r)}{r} = -\frac{\delta(\mathbf{r})}{\epsilon_o}$$

which reduces, for $r \neq 0$, to

$$\nabla^2(\frac{f(r)}{r}) + k^2\frac{f(r)}{r} = 0.$$

– Since (as shown in HW) we have, by using spherical coordinates (reviewed next lecture),

$$\nabla^2(\frac{f(r)}{r}) = \frac{1}{r}\frac{\partial^2 f}{\partial r^2},$$

it follows that we have, for $r \neq 0$,

$$\frac{1}{r}(\frac{\partial^2 f}{\partial r^2} + k^2 f) = 0,$$

which is in turn satisfied by

$$f(r) = ge^{\mp jkr} = ge^{\mp j\omega r/c}$$

with an arbitrary constant g.

– Finally, the constraint that $f(r) = 1/4\pi\epsilon_o$ for $\omega = 0$ indicates that

$$g = \frac{1}{4\pi\epsilon_o},$$

Forced wave eqn (phasor form):

$$\nabla^2\tilde{V} + k^2\tilde{V} = -\frac{\tilde{\rho}}{\epsilon_o}$$

with

$$k = \omega\sqrt{\mu_o\epsilon_o} = \frac{\omega}{c}.$$

and thus

$$f(r) = \frac{e^{\mp jkr}}{4\pi\epsilon_o}$$

in the solutions $f(r)/r$ of the wave equation with $\tilde{\rho}(\mathbf{r}) = \delta(\mathbf{r})$.

This concludes our proof of the postulated solution

$$\tilde{V}(\mathbf{r}) = \frac{e^{-jk|\mathbf{r}|}}{4\pi\epsilon_o|\mathbf{r}|} \quad \text{with} \quad k \equiv \omega\sqrt{\mu_o\epsilon_o} = \frac{\omega}{c}$$

where the sign choice in the exponent favors the physically relevant *causal* solution as opposed to the acausal alternative (see discussion below).

The choice $-jkr$ leads to so-called *retarded* solution of the wave equation. The alternative choice $+jkr$ is not used because it leads to an *advanced* solution that depends on future values of the charge distribution not available in practice (this *causality* constaint is further discussed later in this lecture).

Note that k is another symbol for wavenumber β. In this and higher level courses in EM and signal processing k is favored over β (for a good number of reasons which will become apparent as we learn more).

For the record, by scaling the result above:

For $\tilde{\rho}(\mathbf{r}) = Q\delta(\mathbf{r})$, the causal solution of the forced wave equation

$$\nabla^2\tilde{V} + k^2\tilde{V} = -\frac{\tilde{\rho}}{\epsilon_o},$$

where $k \equiv \omega\sqrt{\mu_o\epsilon_o}$ is the phasor

$$\tilde{V}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_o}\frac{e^{-jkr}}{r}.$$

Likewise, for $\tilde{J}_z(\mathbf{r}) = P\delta(\mathbf{r})$, the causal solution of the forced wave equation

$$\nabla^2\tilde{A}_z + k^2\tilde{A}_z = -\mu_o\tilde{J}_z,$$

where $k \equiv \omega\sqrt{\mu_o\epsilon_o}$ must be the phasor

$$\tilde{A}_z(\mathbf{r}) = \frac{\mu_o P}{4\pi}\frac{e^{-jkr}}{r},$$

which describes, with $P = I\Delta z$, the vector potential of the *Hertzian dipole* defined in Lecture 6.

- We can next argue as follows:

$$\delta(\mathbf{r}) \rightarrow \boxed{\text{Forced Wave Eqn}} \rightarrow \frac{e^{-jk|\mathbf{r}|}}{4\pi\epsilon_o|\mathbf{r}|}$$

  and

$$\delta(\mathbf{r}-\mathbf{r}') \rightarrow \boxed{\text{Forced Wave Eqn}} \rightarrow \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|}$$

  imply that

$$\int \tilde{\rho}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}' = \tilde{\rho}(\mathbf{r}) \rightarrow \boxed{\text{Forced Wave Eqn}} \rightarrow \int \frac{\tilde{\rho}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' = \tilde{V}(\mathbf{r}),$$

  giving us, on the right-hand side, the **retarded potential solution** in the *frequency domain.*

- Finally, inverse Fourier transforming the above result back to time domain, we obtain

$$V(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}',$$

  where we made an explicit use of the time-shift property of the Fourier transform as in

$$\rho(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}) \leftrightarrow R(\mathbf{r}',\omega)e^{-j\omega|\mathbf{r}-\mathbf{r}'|/c} \equiv \tilde{\rho}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}.$$

- Note that $V(\mathbf{r}, t)$ is a weighted superposition of the *past* values of charge density $\rho(\mathbf{r}, t)$ (as opposed to future values) because of our use of the causal solution[1] (as opposed to acausal solution) of the forced wave equation discussed above.

Question: is *causality* an additional postulate on top of Maxwell's equations that needs to be invoked to understand radiation?

Answer: no, not really, we need to invoke causality at this stage to pick the relevant root of the solution for the forced wave equation simply because we took a shortcut of using a steady-state solution based on Fourier transforms (phasors). Had we solved the same problem as an initial value problem (using the Laplace transform), only the retarded potential solution would have figured in our answer naturally without having to invoke a separate causality postulate — see *J. L. Anderson*, "Why we use retarded potentials", *Am.J. Phys.*, 60, 465, 1992.

It is useful to stress at this point the relationship between a phasor (of a time harmonic function) and a Fourier transform (of a time domain function) as follows:

- A phasor, say, $\tilde{V}(\mathbf{r})$ is a sample of a Fourier transform function $V(\mathbf{r}, \omega)$ at the frequency ω of a time-harmonic function that the phasor represents.

- Conversely, a Fourier transform $V(\mathbf{r}, \omega)$ represents a continuous collection of phasors $\tilde{V}(\mathbf{r})$ representing time-harmonic functions of all possible ω.

  Based on the above correspondence principle we feel free to switch between phasor and Fourier transform concepts as convenient.

[1]This choice is also referred to as Sommerfeld's *radiation condition* after Arnold Sommerfeld who also developed an asymptotic formula that retains the causal solution and rejects the acausal one.

Having finished the derivation of the retarded potential solution of the forced wave equation for scalar potential, we can re-state our result, and by *analogy* the result for the retarded vector potential as:

$$V(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}',$$

the solution of inhomogeneous wave equation

$$\nabla^2 V - \mu_o\epsilon_o \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_o}$$

$$\mathbf{A}(\mathbf{r},t) = \int \frac{\mu_o \mathbf{J}(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}',$$

the solution of inhomogeneous wave equation

$$\nabla^2 \mathbf{A} - \mu_o\epsilon_o \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_o \mathbf{J}$$

where

$$c \equiv \frac{1}{\sqrt{\mu_o\epsilon_o}} \text{ is the speed of light in free space.}$$

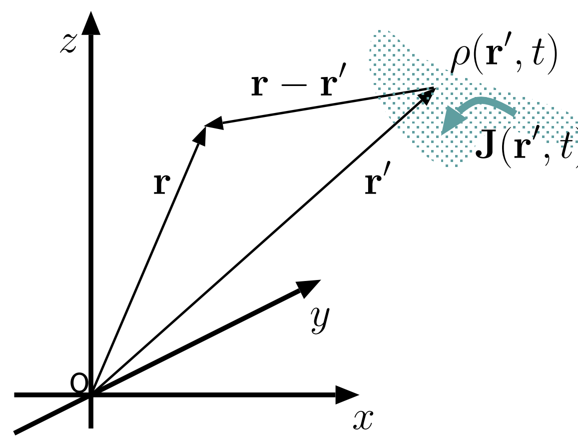
These results indicate that *retarded* potentials

$$V(\mathbf{r},t) \quad \text{and} \quad \mathbf{A}(\mathbf{r},t)$$

are appropriately weighted and *delayed* sums of

$$\rho(\mathbf{r},t) \quad \text{and} \quad \mathbf{J}(\mathbf{r},t)$$

in convolution-like 3D space integrals.



Next turning our attention to retarded **vector potential** solutions, we note that the results stated in time and frequency domains are as follows:

Time-domain:	**Frequency-domain:**
$$\mathbf{A}(\mathbf{r},t) = \int \frac{\mu_o \mathbf{J}(\mathbf{r}', t - \frac{\lvert\mathbf{r}-\mathbf{r}'\rvert}{c})}{4\pi\lvert\mathbf{r}-\mathbf{r}'\rvert} d^3\mathbf{r}',$$ the solution of the inhomogeneous wave equation $$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_o\mathbf{J}.$$	$$\tilde{\mathbf{A}}(\mathbf{r}) = \int \frac{\mu_o \tilde{\mathbf{J}}(\mathbf{r}') e^{-jk\lvert\mathbf{r}-\mathbf{r}'\rvert}}{4\pi\lvert\mathbf{r}-\mathbf{r}'\rvert} d^3\mathbf{r}',$$ the solution of the inhomogeneous wave equation $$\nabla^2\tilde{\mathbf{A}} + \frac{\omega^2}{c^2}\tilde{\mathbf{A}} = -\mu_o\tilde{\mathbf{J}}.$$

In the next lecture we will learn how to perform vector calculus operations in spherical coordinates and then apply the frequency-domain result obtained above to the calculation of radiation from short current elements.