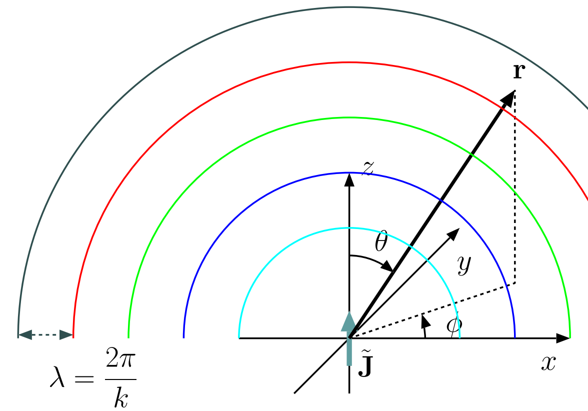
7 Hertzian dipole fields

- We concluded last lecture with the retarded potential solutions



Frequency-domain:

$$\tilde{\mathbf{A}}(\mathbf{r}) = \frac{\mu_o}{4\pi} I\Delta z \frac{e^{-jkr}}{r}\hat{z}$$

Time-domain:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi} I\Delta z \frac{\cos(\omega t - kr)}{r}\hat{z}$$

of a $\hat{z}$ directed Hertzian dipole.

- We noted that these oscillatory solutions describe spherical waves by virtue of the e^{-jkr} dependence of the potential phasor on r:

  – the variable r measures distance in *all* directions away from the origin, as opposed to, say, x measuring distance only along one coordinate axis labelled as x.

  Thus, while the phasor variation e^{-jkx} describes a plane wave, the phasor e^{-jkr} describes a spherical wave (see margin).

We will next determine the magnetic and electric fields produced by a Hertzian dipole.

- To calculate the magnetic field phasor $\tilde{\mathbf{B}}$ we will make use of

$$\tilde{\mathbf{B}} = \nabla\times\tilde{\mathbf{A}} \quad \text{and} \quad \nabla\times\tilde{\mathbf{A}} = \begin{vmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \tilde{A}_r & r\tilde{A}_\theta & r\sin\theta\tilde{A}_\phi \end{vmatrix} \text{ in spherical coordinates.}$$

- Given that

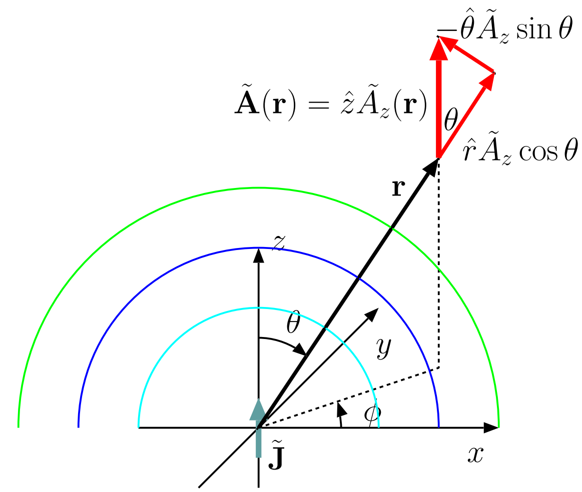
$$\tilde{\mathbf{A}}(\mathbf{r}) = \underbrace{\frac{\mu_o}{4\pi} I\Delta z \frac{e^{-jkr}}{r}}_{\tilde{A}_z(\mathbf{r})} \hat{z}$$

and

$$\hat{z}\cdot\hat{r} = \cos\theta, \quad \hat{z}\cdot\hat{\theta} = -\sin\theta, \quad \hat{z}\cdot\hat{\phi} = 0,$$

it follows that

$$\begin{aligned} \tilde{A}_r &= \tilde{\mathbf{A}}(\mathbf{r})\cdot\hat{r} = \tilde{A}_z(\mathbf{r})\cos\theta, \\ \tilde{A}_\theta &= \tilde{\mathbf{A}}(\mathbf{r})\cdot\hat{\theta} = -\tilde{A}_z(\mathbf{r})\sin\theta, \\ \tilde{A}_\phi &= \tilde{\mathbf{A}}(\mathbf{r})\cdot\hat{\phi} = 0. \end{aligned}$$



$\tilde{A}_r(\mathbf{r}) = \tilde{A}_z(\mathbf{r})\cos\theta$

$\tilde{A}_\theta(\mathbf{r}) = -\tilde{A}_z(\mathbf{r})\sin\theta$

Substituting $\tilde{A}_r$, $\tilde{A}_\theta$, $\tilde{A}_\phi$ into the curl formula, we proceed as

$$\nabla\times\tilde{\mathbf{A}} = \frac{\mu_o}{4\pi} I\Delta z \begin{vmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{e^{-jkr}}{r}\cos\theta & -r\frac{e^{-jkr}}{r}\sin\theta & 0 \end{vmatrix}.$$

Expanding the determinant, we obtain

$$\begin{aligned}\tilde{\mathbf{B}} = \nabla \times \tilde{\mathbf{A}} &= \frac{\mu_o}{4\pi} I\Delta z \frac{\hat{\phi}}{r}\{-\frac{\partial}{\partial r} e^{-jkr} \sin\theta - \frac{\partial}{\partial \theta}\frac{e^{-jkr}}{r}\cos\theta\} \\ &= \frac{\mu_o}{4\pi} I\Delta z (jk + \frac{1}{r}) \sin\theta \frac{e^{-jkr}}{r}\hat{\phi}.\end{aligned}$$

Consequently,

$$\tilde{\mathbf{H}} = jkI\Delta z(1 + \frac{1}{jkr}) \sin\theta \frac{e^{-jkr}}{4\pi r}\hat{\phi}.$$

Notice, the wave field

$$\tilde{\mathbf{H}}(\mathbf{r}) = \hat{\phi}\tilde{H}_\phi(\mathbf{r})$$

of the Hertzian dipole is purely "azimuthal" — this is the direction the *right-hand-rule* would give if the right-hand-thumb were directed in the direction of dipole current.

- To obtain the accompanying electric field phasor we will next employ Ampere's law

$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\epsilon_o \tilde{\mathbf{E}},$$

with $\tilde{\mathbf{J}} = 0$, which is true at all locations outside the Hertzian dipole. In that case

$$\tilde{\mathbf{E}} = \frac{\nabla \times \tilde{\mathbf{H}}}{j\omega\epsilon_o} =$$

$$= \frac{1}{j\omega\epsilon_o}\begin{vmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \tilde{H}_r & r\tilde{H}_\theta & r\sin\theta\tilde{H}_\phi \end{vmatrix} = \frac{1}{j\omega\epsilon_o}\begin{vmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta\tilde{H}_\phi \end{vmatrix}$$

$$= \frac{1}{j\omega\epsilon_o}\{\frac{\hat{r}}{r^2\sin\theta}\frac{\partial}{\partial\theta} r\sin\theta\tilde{H}_\phi - \frac{\hat{\theta}}{r\sin\theta}\frac{\partial}{\partial r} r\sin\theta\tilde{H}_\phi\}$$

$$= \frac{1}{j\omega\epsilon_o}\{\frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\tilde{H}_\phi - \frac{\hat{\theta}}{r}\frac{\partial}{\partial r}r\tilde{H}_\phi\}.$$

Substituting

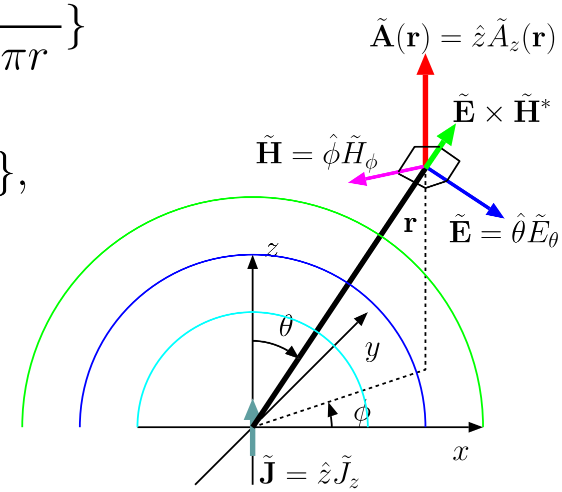
$$\tilde{H}_\phi = jkI\Delta z(1+\frac{1}{jkr})\sin\theta\frac{e^{-jkr}}{4\pi r}$$

from above, and simplifying, we have

$$\begin{aligned}
\tilde{\mathbf{E}} &= \frac{jkI\Delta z}{j\omega\epsilon_o}\{\frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta(1+\frac{1}{jkr})\sin\theta\frac{e^{-jkr}}{4\pi r} - \frac{\hat{\theta}}{r}\frac{\partial}{\partial r}r(1+\frac{1}{jkr})\sin\theta\frac{e^{-jkr}}{4\pi r}\} \\
&= \frac{kI\Delta z}{\omega\epsilon_o}\{\frac{\hat{r}}{r\sin\theta}(1+\frac{1}{jkr})\frac{e^{-jkr}}{4\pi r}\frac{\partial}{\partial\theta}\sin^2\theta - \frac{\hat{\theta}}{r}\sin\theta\frac{\partial}{\partial r}(1+\frac{1}{jkr})\frac{e^{-jkr}}{4\pi}\} \\
&= \sqrt{\frac{\mu_o}{\epsilon_o}}I\Delta z\{\frac{\hat{r}}{r}(1+\frac{1}{jkr})\frac{e^{-jkr}}{4\pi r}2\cos\theta + \hat{\theta}\sin\theta[jk(1+\frac{1}{jkr}) + \frac{1}{jkr^2}]\frac{e^{-jkr}}{4\pi r}\} \\
&= jk\eta_o I\Delta z\frac{e^{-jkr}}{4\pi r}\{\hat{r}[\frac{1}{jkr} + (\frac{1}{jkr})^2]2\cos\theta + \hat{\theta}[1 + \frac{1}{jkr} + (\frac{1}{jkr})^2]\sin\theta\},
\end{aligned}$$

where

$$\eta_o \equiv \sqrt{\frac{\mu_o}{\epsilon_o}}.$$



- This is a very complicated looking result.
  - Fortunately, many of the terms above are important only at very small values of r!

- If were to drop all of the terms in $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ above except for those varying as $\frac{1}{r}$, we would be left with

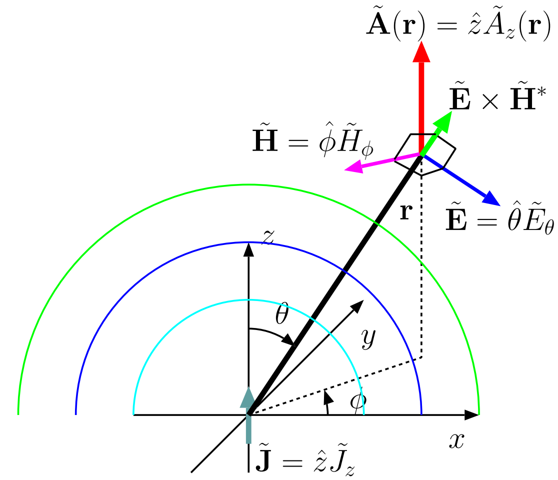
$$\tilde{\mathbf{E}} = j\eta_o Ik\Delta z \sin\theta \frac{e^{-jkr}}{4\pi r}\hat{\theta} \quad \text{and} \quad \tilde{\mathbf{H}} = jIk\Delta z \sin\theta \frac{e^{-jkr}}{4\pi r}\hat{\phi},$$

which are the only terms of the fields of the Hertzian dipole that matter at large distances (of interest for communication and remote sensing purposes).

  - They are called the **radiation fields** of the Hertzian dipole, and the remainder (the terms which have been dropped) are called the **storage fields**.
  - The reasoning behind this terminology is as follows:
    The average Poynting vector

$$\langle \mathbf{E} \times \mathbf{H} \rangle = \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\}$$

    computed with the full expressions for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ gives the same result as that computed with only the simplified *radiation* fields.
  - What that means is the remaining parts of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ (*storage* fields) do not contribute to the transport of energy away from the dipole.
  - They only represent a local energy exchange (and storage) between inductive and capacitive attributes of the dipole — recall that the



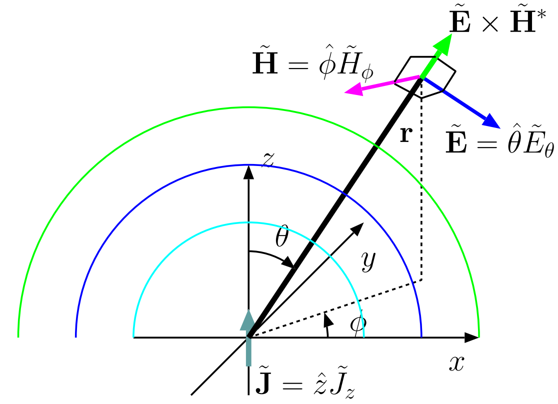
Radiation fields:

$$\tilde{\mathbf{E}} = j\eta_o Ik\Delta z \sin\theta \frac{e^{-jkr}}{4\pi r}\hat{\theta}$$

and

$$\tilde{\mathbf{H}} = jIk\Delta z \sin\theta \frac{e^{-jkr}}{4\pi r}\hat{\phi}.$$

dipole is both a filament having some inductance and a capacitor with two reservoirs for charge storage.



In many applications of radiation theory we only need to focus on the radiation fields.

Fortunately, the expressions for radiation fields are simple and have features resembling the plane TEM waves that we are already familiar with. Let's see what these features are:

Radiation fields:

$$\tilde{\mathbf{E}} = j\eta_o I k\Delta z \sin\theta \frac{e^{-jkr}}{4\pi r}\hat{\theta}$$

and

$$\tilde{\mathbf{H}} = jIk\Delta z \sin\theta \frac{e^{-jkr}}{4\pi r}\hat{\phi}.$$

1. The phasors are **orthogonal** and

$$\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \propto \hat{\theta} \times \hat{\phi} = \hat{r}$$

points in the radial direction $\hat{r}$ of the spherical wave propagation just as in plane TEM waves.

2. The magnitude of $\tilde{\mathbf{H}}$ can be obtained by dividing the magnitude of $\tilde{\mathbf{E}}$ by the intrinsic impedance η_o just as for plane TEM waves.

3. Conversely, the magnitude of $\tilde{\mathbf{E}}$ can be obtained by multiplying the magnitude of $\tilde{\mathbf{H}}$ by the intrinsic impedance η_o just as for plane TEM waves.

4. The direction of $\tilde{\mathbf{H}}$ can be deduced from the direction of $\tilde{\mathbf{E}}$ (and vice versa) by a 90° rotation and enforcing the right-hand-rule of having $\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*$ point in $\hat{r}$ direction.

On the other hand, these spherical TEM waves radiated by the Hertzian dipole differ from *uniform* plane TEM waves by the facts that:

1. Field amplitude is not constant in the propagation direction because of $\frac{1}{r}$ dependence.

2. Field amplitude is not constant in the direction orthogonal to the propagation direction because of $\sin\theta$ dependence.

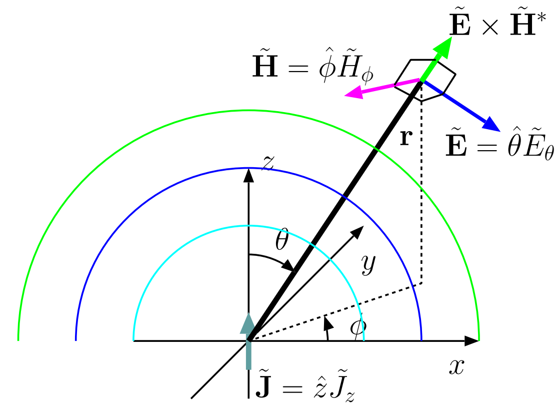
   As such, a Hertzian dipole radiates TEM waves which are *non-uniform* as well as spherical (non-planar).

   As such, Hertzian dipole radiation is said to be **anisotropic**!

- Radiation is strong — forms a "beam", so to speak — in the **broadside direction** of $\theta = 90°$ (with respect to the dipole axis),

- Radiation vanishes for $\theta = 0°, 180°$ along the **dipole axis**.

  – In short, radiation strength scales with $\Delta z \sin\theta$, a foreshortened version of length Δz of the dipole "seen" from an angle θ with respect to the dipole axis. More on this later on...



Radiation fields:

$$\tilde{\mathbf{E}} = j\eta_o I k\Delta z \sin\theta \frac{e^{-jkr}}{4\pi r}\hat{\theta}$$

and

$$\tilde{\mathbf{H}} = jIk\Delta z \sin\theta \frac{e^{-jkr}}{4\pi r}\hat{\phi}.$$