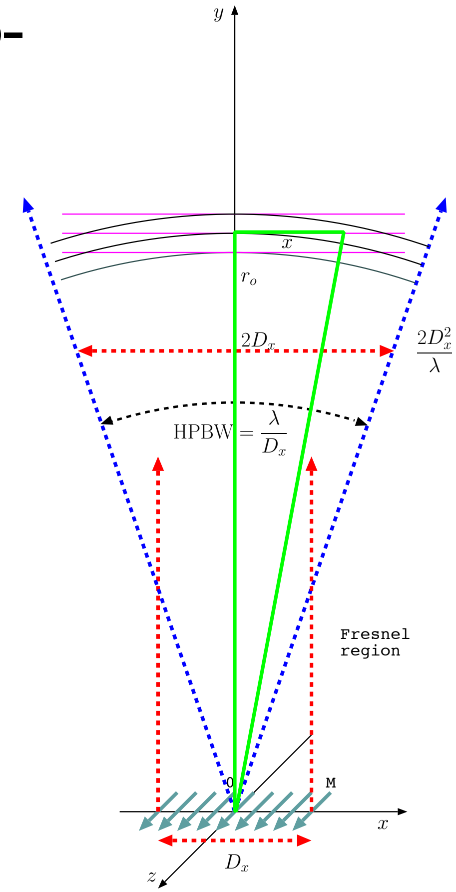
15 Plane-wave form of Maxwell's equations, propagation in arbitrary direction



Having seen how EM waves are generated by radiation sources and how spherical TEM waves develop a "planar" character over increasingly large regions as they propagate away from their sources, it is time to shift our attention to *propagation* and *guidance phenomena* using a plane-wave formalism.

Perhaps the most "practical" rationalization of this switch from spherical to plane-wave emphasis is that waves produced by compact sources invariably "look" planar at the scales of practical receiving systems (that will study near the end of this course) situated afar.

- We wish to study wave solutions of Maxwell's equations exhibiting the planar phasor form

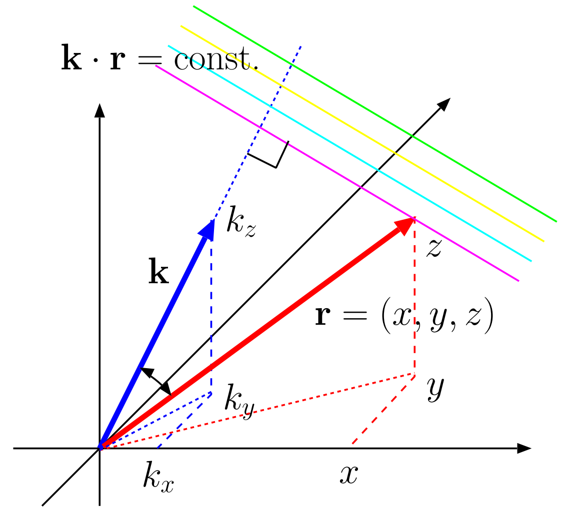
  $$\tilde{\mathbf{E}} = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}} = \hat{e} E_o e^{-j\mathbf{k}\cdot\mathbf{r}}$$

  and time-domain variations

  $$\begin{aligned} \text{Re}\{\tilde{\mathbf{E}} e^{j\omega t}\} &= \text{Re}\{\mathbf{E}_o e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}\} \\ &= \hat{e}|E_o|\cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \angle E_o) \end{aligned}$$

  where **wave vector k** is to be found in compliance with ω and Maxwell's equations according to some specific "dispersion relation" including the details of the propagation medium.

- For simplicity, the above phasor has been declared to be linearly polarized. Circular or elliptic polarized wave fields can be constructed later on via superposition methods.



- Linearly polarized wave field phasor above can be expanded as

$$\tilde{\mathbf{E}} = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}} = \mathbf{E}_o e^{-j(k_x x + k_y y + k_z z)}$$

assuming a wave vector

$$\mathbf{k} = (k_x, k_y, k_z) = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$$

expressed in terms of its projections (k_x, k_y, k_z) along the Cartesian coordinate axes (x, y, z).

- A special case we are familiar with is

$$k_x = k_y = 0,\ k_z > 0,\ \text{ when } \mathbf{k} = k_z\hat{z} = k\hat{z} \ \text{ and } e^{-j\mathbf{k}\cdot\mathbf{r}} = e^{-jkz}$$

as in plane TEM waves travelling in $+z$ direction having a

$$\text{wavelength } \lambda = \frac{2\pi}{k} \text{ and propagation speed } v_p = \frac{\omega}{k}.$$

  - Likewise, the case

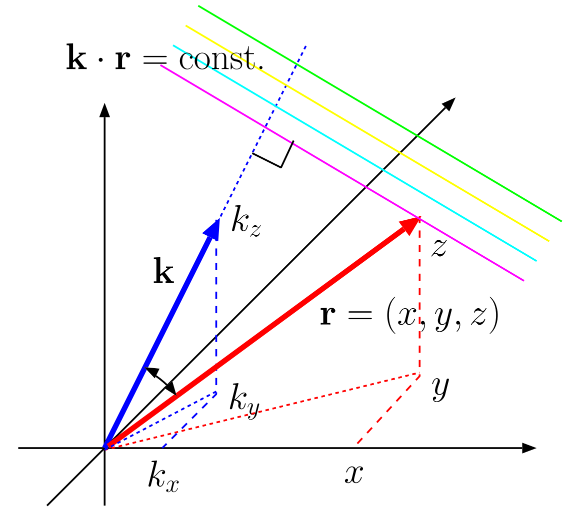
$$k_y = k_z = 0\,, k_x > 0,\ \text{ when } \mathbf{k} = k_x\hat{x} = k\hat{x} \ \text{ and } e^{-j\mathbf{k}\cdot\mathbf{r}} = e^{-jkx}$$

  corresponds to plane TEM waves travelling in $+x$ direction with the same wavelength and propagation speed.

- The general case with non-zero components (k_x, k_y, k_z) corresponds to a plane wave propagating in the direction of unit vector

$$\hat{k} \equiv \frac{\mathbf{k}}{k} = \frac{(k_x, k_y, k_z)}{k} \text{ where } k \equiv |\mathbf{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{2\pi}{\lambda}$$

and also having the same wavelength and propagation speed as above. Wavelength λ now describes the shift invariance of the wave field in spatial $\hat{k}$ direction, i.e., the propagation direction.



Example 1: A plane wave electric field phasor is specified as

$$\tilde{\mathbf{E}} = \hat{z} e^{-j(3\pi x - 4\pi y)} \frac{\text{V}}{\text{m}}.$$

Determine the propagation direction $\hat{k}$, wavenumber $k = |\mathbf{k}|$, wavelength $\lambda = \frac{2\pi}{k}$ and wave frequency $f = \frac{\omega}{2\pi}$ assuming a propagation speed $c = 3 \times 10^8$ m/s.

Solution: Contrasting $\tilde{\mathbf{E}}$ with $e^{-j(k_x x + k_y y + k_z z)}$, we note that

$$k_x = 3\pi \frac{\text{rad}}{\text{m}}, \quad k_y = -4\pi \frac{\text{rad}}{\text{m}}, \quad k_z = 0.$$

Hence, wave vector

$$\mathbf{k} = \hat{x} k_x + \hat{y} k_y + \hat{z} k_z = 3\pi \hat{x} - 4\pi \hat{y} \frac{\text{rad}}{\text{m}},$$

and wave number

$$k = |\mathbf{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{(3\pi)^2 + (4\pi)^2 + 0^2} = \sqrt{25\pi^2} = 5\pi \frac{\text{rad}}{\text{m}}.$$

The propagation direction is specified by the unit vector

$$\hat{k} = \frac{\mathbf{k}}{k} = \frac{3\pi\hat{x} - 4\pi\hat{y}}{5\pi} = 0.6\hat{x} - 0.8\hat{y}.$$

The wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{5\pi} = 0.4\,\mathrm{m}.$$

Since

$$c = v_p = \frac{\omega}{k}$$

in general, it follows that

$$\omega = kc = 5\pi \times 3 \times 10^8 = 2\pi \times 7.5 \times 10^8\,\frac{\mathrm{rad}}{\mathrm{s}}$$

and

$$f = \frac{\omega}{2\pi} = 750 \times 10^6\,\mathrm{Hz} = 750\,\mathrm{MHz}.$$

- Based on what we learned in ECE 329, we recognize that the wave analyzed in Example 1 must have been propagating in free space.

- What are the constraints on wave vector $\mathbf{k}$ for plane waves propagating in arbitrary media?

To answer the above question, we will return to macroscopic-form Maxwell's equations written in phasor form (see margin) and examine under which conditions phasor solutions

$$\propto e^{-j\mathbf{k}\cdot\mathbf{r}}$$

can be applicable for all the field quantities in the absence of source currents $\tilde{\mathbf{J}}$ and their accompanying $\tilde{\rho}$.

- First, we note that in view of relation

$$\tilde{\mathbf{D}} = \epsilon\tilde{\mathbf{E}},$$

  we can have plane-wave solutions of the form

$$\tilde{\mathbf{D}} = \mathbf{D}_o e^{-j\mathbf{k}\cdot\mathbf{r}} \text{ and } \tilde{\mathbf{E}} = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}}$$

  if and only if ϵ does not depend on position $\mathbf{r}$ (why?).

- Likewise, relation

$$\tilde{\mathbf{B}} = \mu\tilde{\mathbf{H}},$$

  implies plane-wave solutions

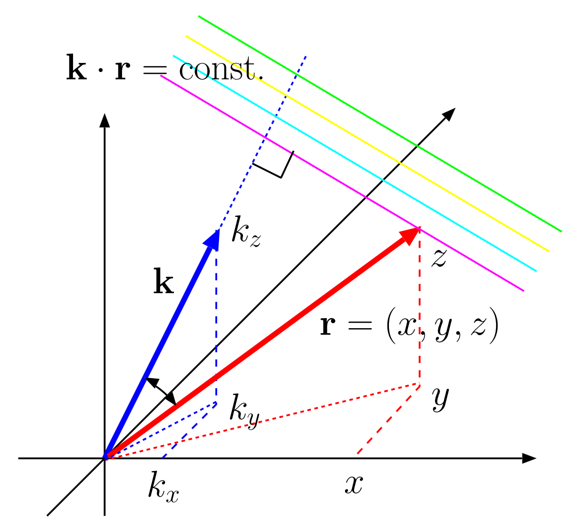
$$\tilde{\mathbf{B}} = \mathbf{B}_o e^{-j\mathbf{k}\cdot\mathbf{r}} \text{ and } \tilde{\mathbf{H}} = \mathbf{H}_o e^{-j\mathbf{k}\cdot\mathbf{r}}$$

  if and only if μ does not depend on position $\mathbf{r}$ (why?).

$$\begin{aligned}\nabla\cdot\tilde{\mathbf{D}} &= \tilde{\rho}\\ \nabla\cdot\tilde{\mathbf{B}} &= 0\\ \nabla\times\tilde{\mathbf{E}} &= -j\omega\tilde{\mathbf{B}}\\ \nabla\times\tilde{\mathbf{H}} &= \tilde{\mathbf{J}} + j\omega\tilde{\mathbf{D}}\end{aligned}$$

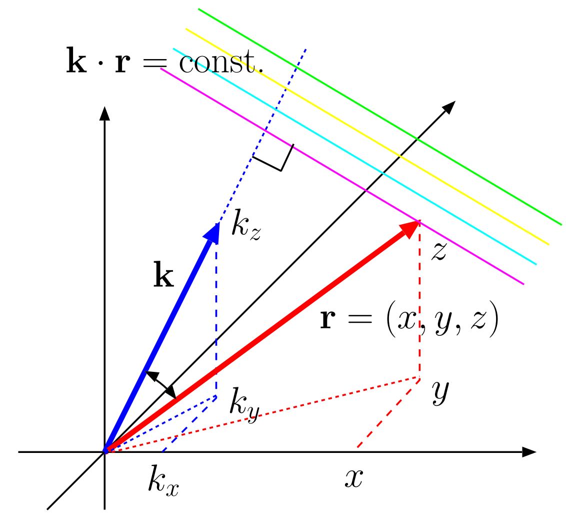
where (constitutive relations)

$$\begin{aligned}\tilde{\mathbf{D}} &= \epsilon\tilde{\mathbf{E}}\\ \tilde{\mathbf{B}} &= \mu\tilde{\mathbf{H}}\\ \tilde{\mathbf{J}}_c &= \sigma\tilde{\mathbf{E}}.\end{aligned}$$



- In a homogeneous region where ϵ, μ, and σ are, by definition, independent of $\mathbf{r}$, plane-wave solutions of phasor-form Maxwell's equations given in the margin become possible provided that

$$\begin{aligned} -j\mathbf{k}\cdot\tilde{\mathbf{D}} &= \tilde{\rho} \\ -j\mathbf{k}\cdot\tilde{\mathbf{B}} &= 0 \\ -j\mathbf{k}\times\tilde{\mathbf{E}} &= -j\omega\tilde{\mathbf{B}} \\ -j\mathbf{k}\times\tilde{\mathbf{H}} &= \tilde{\mathbf{J}} + j\omega\tilde{\mathbf{D}}. \end{aligned}$$



We have obtained these vector-algebraic relations from phasor-form Maxwell's equations in the margin after replacing the vector-differential operator ∇ by the vector-algebraic operator $-j\mathbf{k}$.

The justification of this simple procedure is as follows:

If

$$\tilde{\mathbf{D}} = \mathbf{D}_o e^{-j\mathbf{k}\cdot\mathbf{r}} = \mathbf{D}_o e^{-j(k_x x + k_y y + k_z z)} = (D_{xo}, D_{yo}, D_{zo}) e^{-j(k_x x + k_y y + k_z z)}$$

then

$$\begin{aligned} \nabla\cdot\tilde{\mathbf{D}} &= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial y}) \cdot (D_{xo} e^{-j(k_x x + k_y y + k_z z)}, D_{yo} e^{-j(k_x x + k_y y + k_z z)}, D_{zo} e^{-j(k_x x + k_y y + k_z z)}) \\ &= -jk_x D_{xo} e^{-j(k_x x + k_y y + k_z z)} - jk_y D_{yo} e^{-j(k_x x + k_y y + k_z z)} - jk_z D_{zo} e^{-j(k_x x + k_y y + k_z z)} \\ &= -j(k_x, k_y, k_z) \cdot (D_{xo}, D_{yo}, D_{zo}) e^{-j(k_x x + k_y y + k_z z)} = -j\mathbf{k}\cdot\tilde{\mathbf{D}}. \end{aligned}$$

Likewise, if

$$\tilde{\mathbf{E}} = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}} = \mathbf{E}_o e^{-j(k_x x + k_y y + k_z z)} = (E_{xo}, E_{yo}, E_{zo}) e^{-j(k_x x + k_y y + k_z z)}$$

then

$$
\begin{aligned}
\nabla\times\tilde{\mathbf{E}} &= (\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial y})\times(E_{xo}e^{-j(k_x x+k_y y+k_z z)},E_{yo}e^{-j(k_x x+k_y y+k_z z)},E_{zo}e^{-j(k_x x+k_y y+k_z z)}) \\
&= (-jk_x,-jk_y,-jk_z)\times(E_{xo}e^{-j(k_x x+k_y y+k_z z)},E_{yo}e^{-j(k_x x+k_y y+k_z z)},E_{zo}e^{-j(k_x x+k_y y+k_z z)}) \\
&= -j\mathbf{k}\times\tilde{\mathbf{E}}.
\end{aligned}
$$

The vector-algebraic relations above, repeated in the margin (after canceling out some common terms), are known as plane-wave form of Maxwell's equations.

- Plane-wave form ME in the margin provide us with the constraints such plane waves satisfy in various types of propagation media categorized according to ϵ, μ, and σ.
- Focusing first on the case $\tilde{\rho}=\tilde{\mathbf{J}}=0$ and $\sigma=0$ (source free and non-conducting), the equations simplify as

$$
\begin{aligned}
\mathbf{k}\cdot\tilde{\mathbf{D}} &= 0 \\
\mathbf{k}\cdot\tilde{\mathbf{B}} &= 0 \\
\mathbf{k}\times\tilde{\mathbf{E}} &= \omega\tilde{\mathbf{B}} \\
-\mathbf{k}\times\tilde{\mathbf{H}} &= \omega\tilde{\mathbf{D}}.
\end{aligned}
$$

The first two constraints tell us that wave vector $\mathbf{k}$ is necessarily orthogonal to both $\tilde{\mathbf{D}}=\epsilon\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}=\mu\tilde{\mathbf{H}}$.

  - Hence, the plane waves satisfying the above equations will be TEM.

Plane-wave form of Maxwell's equations:

$$
\begin{aligned}
-j\mathbf{k}\cdot\tilde{\mathbf{D}} &= \tilde{\rho} \\
\mathbf{k}\cdot\tilde{\mathbf{B}} &= 0 \\
\mathbf{k}\times\tilde{\mathbf{E}} &= \omega\tilde{\mathbf{B}} \\
-j\mathbf{k}\times\tilde{\mathbf{H}} &= \tilde{\mathbf{J}}+j\omega\tilde{\mathbf{D}}.
\end{aligned}
$$

- Cross-multiplying the third equation with $\mathbf{k}$ and substituting from the fourth equation we get

$$\mathbf{k}\times(\mathbf{k}\times\tilde{\mathbf{E}})=\omega\mu\mathbf{k}\times\tilde{\mathbf{H}}=\omega\mu(-\omega\tilde{\mathbf{D}})=-\mu\epsilon\omega^2\tilde{\mathbf{E}}.$$

But then

$$\mathbf{k}\times(\mathbf{k}\times\tilde{\mathbf{E}})=-|\mathbf{k}|^2\tilde{\mathbf{E}}$$

since vectors $\mathbf{k}$ and $\tilde{\mathbf{E}}$ are perpendicular as shown in the margin — cross-multiplying $\tilde{\mathbf{E}}$ twice by $\mathbf{k}$ produces $-\tilde{\mathbf{E}}$ times the magnitude square of $\mathbf{k}$, i.e. $\mathbf{k}\cdot\mathbf{k}=|\mathbf{k}|^2$!

  - The above lines are compatible if and only if

$$\mathbf{k}\cdot\mathbf{k}=\omega^2\mu\epsilon \quad\Rightarrow\quad k\equiv|\mathbf{k}|=\omega\sqrt{\mu\epsilon}.$$

  Hence, plane-wave solutions

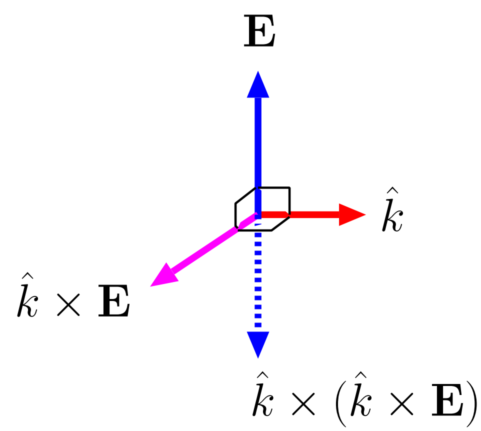
$$\propto e^{-j\omega\sqrt{\mu\epsilon}\hat{k}\cdot\mathbf{r}}$$

  are allowed as long as

$$\hat{k}\cdot\tilde{\mathbf{E}}=0 \quad\text{and}\quad \hat{k}\cdot\tilde{\mathbf{H}}=0.$$

  Furthermore, according to the last two equations in the margin,

$$\tilde{\mathbf{H}}=\frac{\hat{k}\times\tilde{\mathbf{E}}}{\eta} \quad\text{and}\quad \tilde{\mathbf{E}}=\eta\tilde{\mathbf{H}}\times\hat{k} \quad\text{with}\quad \eta=\sqrt{\frac{\mu}{\epsilon}}.$$



Also, the vector identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{C}\cdot\mathbf{A})\mathbf{B}-(\mathbf{B}\cdot\mathbf{A})\mathbf{C}$$

leads to the same result.

Plane-wave form of Maxwell's equations:

$$\begin{aligned}\mathbf{k}\cdot\tilde{\mathbf{D}}&=0\\ \mathbf{k}\cdot\tilde{\mathbf{B}}&=0\\ \mathbf{k}\times\tilde{\mathbf{E}}&=\omega\mu\tilde{\mathbf{H}}\\ -\mathbf{k}\times\tilde{\mathbf{H}}&=\omega\epsilon\tilde{\mathbf{E}}.\end{aligned}$$