16 Reflection and transmission, TE mode

- Last lecture we learned how to represent plane-TEM waves propagating in a direction $\hat{k}$ in terms of field phasors

$$\tilde{\mathbf{E}} = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}} \text{ and } \tilde{\mathbf{H}} = \frac{\hat{k} \times \tilde{\mathbf{E}}}{\eta}$$

such that

$$\eta = \sqrt{\frac{\mu}{\epsilon}}, \quad \mathbf{k} = k\hat{k}, \quad k = \omega\sqrt{\mu\epsilon}, \text{ and } \mathbf{k}\cdot\mathbf{E}_o = 0.$$

Such waves are only permitted in **homogeneous** propagation media with constant μ and ϵ and zero σ.

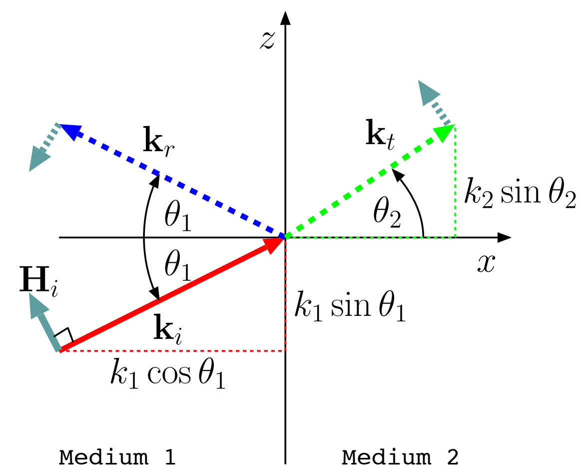
  - The condition of zero σ can be relaxed easily — in that case the above relations would still hold if we were to replace ϵ by $\epsilon + \frac{\sigma}{j\omega}$ as we will see later on.

- In this lecture we will examine the propagation of plane-TEM waves across two distinct homogeneous media having a planar interface between them.

- With no loss of generality we can choose unit vector $\hat{x}$ be the unit-normal of the interface plane separating **medium 1 in the region** $x < 0$ from **medium 2 in the region** $x > 0$.

In 1808 Etienne-Loius *Malus* discovered that light reflected from a surface at an oblique angle will in general be *polarized* differently than the incident wave on the reflecting surface.

This is caused by the difference of the reflection coefficients of TE and TM components of the incident wave as we will learn in this lecture. Practical implementation of the phenomenon include polarizers and polarizing filters used in optical instruments, photography, and LCD displays.



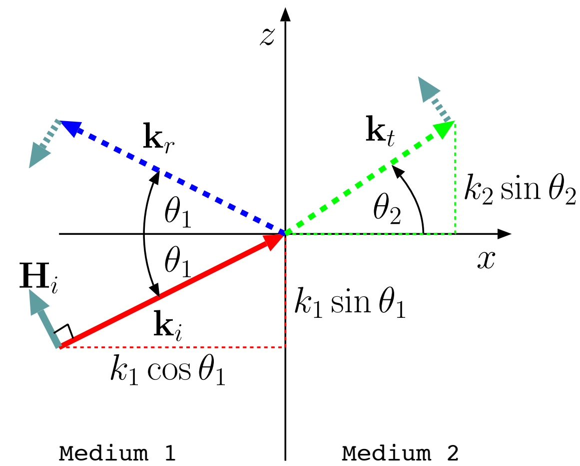
- A plane-TEM wave incident onto the interface from medium 1 is assigned a wavevector

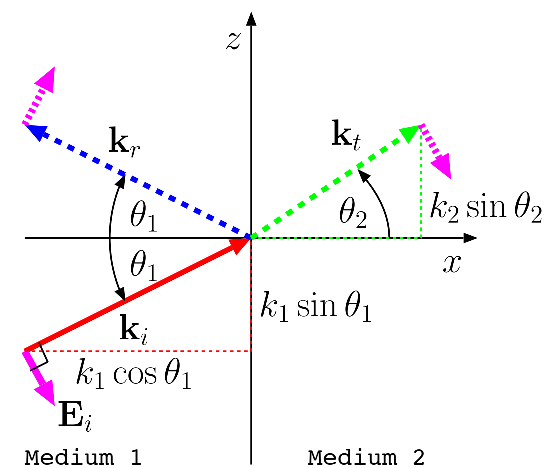
$$\mathbf{k}_i = k_1(\hat{x}\cos\theta_1 + \hat{z}\sin\theta_1)$$

by taking $\hat{y}$ to be orthogonal to $\mathbf{k}_i$ (see margin).

This makes the xz-plane the "plane of incidence" and θ_1 the "angle of incidence", and, furthermore,

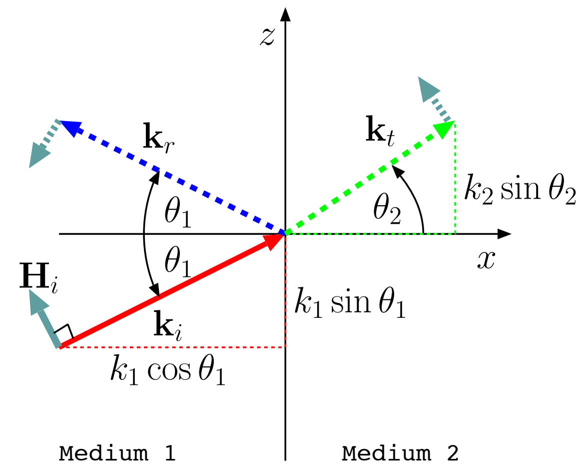
  - if we were to consider the case of $\tilde{\mathbf{E}}_i = \hat{y}E_o e^{-j\mathbf{k}_i\cdot\mathbf{r}}$ we would call the problem a "TE mode" problem, where TE is short for **T**ransverse **E**lectric field, and transverse is with respect to the plane of incidence.
  - if we were to consider the case of $\tilde{\mathbf{H}}_i = \hat{y}H_o e^{-j\mathbf{k}_i\cdot\mathbf{r}}$ we would call the problem a "TM mode" problem, where TM is short for **T**ransverse **M**agnetic field, and transverse is, once again, with respect to the plane of incidence.





This lecture we will examine the TE mode, and next lecture the TM mode. These different modes have different transmission and reflection properties. They are easy to study one at a time, and sufficient in general since all cases can be represented as a superposition of TE and TM cases.

TE mode reflection problem:



- In TE mode reflection problem, the plane-wave field phasors incident on the interface between medium 1 and 2 — $x = 0$ plane — are specified as

$$\tilde{\mathbf{E}}_i = \hat{y} E_o e^{-j\mathbf{k}_i \cdot \mathbf{r}} \text{ and } \tilde{\mathbf{H}}_i = \frac{\mathbf{k}_i \times \tilde{\mathbf{E}}_i}{k_1 \eta_1},$$

  where

$$\mathbf{k}_i = k_1(\hat{x}\cos\theta_1 + \hat{z}\sin\theta_1),$$

  and

$$k_1 = \frac{\omega}{v_1},\ v_1 = \frac{1}{\sqrt{\mu_1 \epsilon_1}},\ \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}.$$

  – The plane-wave field specified above satisfies Maxwell's equations in the homogeneous medium 1 occupying the region $x < 0$, but if there were no other fields in media 1 and 2, *Maxwell's boundary condition equations* requiring the continuity of tangential $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ across any boundary not supporting a surface current would be violated.

In order to comply with the boundary condition equations we postulate a set of *reflected* and *transmitted* wave fields in media 1 and 2 as follows:

- In medium 1 we postulate a **reflected plane-wave** with field phasors

$$\tilde{\mathbf{E}}_r = \hat{y} E_o \Gamma_\perp e^{-j\mathbf{k}_r \cdot \mathbf{r}} \text{ and } \tilde{\mathbf{H}}_r = \frac{\mathbf{k}_r \times \tilde{\mathbf{E}}_r}{k_1 \eta_1},$$

where

$$\mathbf{k}_r = k_1(-\hat{x}\cos\theta_1 + \hat{z}\sin\theta_1).$$

- In medium 2 we postulate a **transmitted plane-wave** with field phasors

$$\tilde{\mathbf{E}}_t = \hat{y}E_o\tau_\perp e^{-j\mathbf{k}_t\cdot\mathbf{r}} \text{ and } \tilde{\mathbf{H}}_t = \frac{\mathbf{k}_t \times \tilde{\mathbf{E}}_t}{k_2\eta_2},$$

where

$$\mathbf{k}_t = k_2(\hat{x}\cos\theta_2 + \hat{z}\sin\theta_2).$$

and

$$k_2 = \frac{\omega}{v_2},\ v_2 = \frac{1}{\sqrt{\mu_2\epsilon_2}},\ \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}.$$

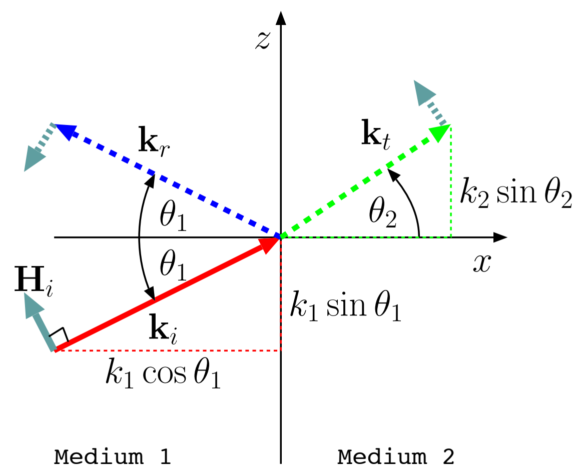
- To justify our postulates and determine a set of *reflection* and *transmission* coefficients $\Gamma_\perp$ and $\tau_\perp$ — defined in terms of electric field components — we will next apply the boundary conditions on $x = 0$ surface, where (using $x = 0$)

$$\tilde{\mathbf{E}}_i = \hat{y}E_o e^{-jk_1\sin\theta_1 z},\ \tilde{\mathbf{E}}_r = \hat{y}E_o\Gamma_\perp e^{-jk_1\sin\theta_1 z},\ \tilde{\mathbf{E}}_t = \hat{y}E_o\tau_\perp e^{-jk_2\sin\theta_2 z}.$$

Clearly, with these field components tangential continuity of the total field phasor $\tilde{\mathbf{E}}$ across $x = 0$ surface will be satisfied for all z if and only if

$$e^{-jk_1\sin\theta_1 z} + \Gamma_\perp e^{-jk_1\sin\theta_1 z} = \tau_\perp e^{-jk_2\sin\theta_2 z},$$

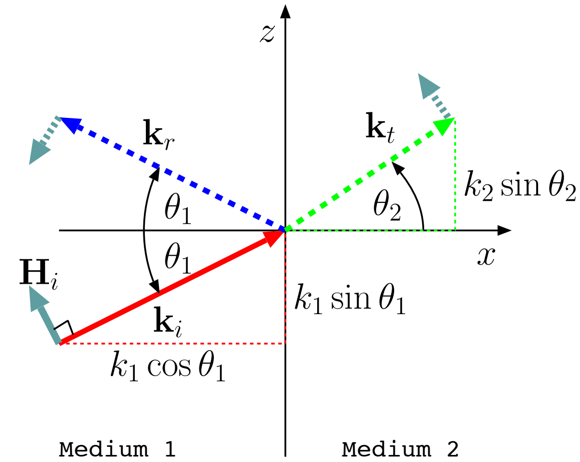
which is only possible (non-trivially) if



1. A "phase matching" condition[1]

$$k_1 \sin\theta_1 = k_2 \sin\theta_2$$

known as **Snell's law** is satisfied, and



2. $\Gamma_\perp$ and $\tau_\perp$ satisfy

$$1 + \Gamma_\perp = \tau_\perp.$$

- Tangential components of $\tilde{\mathbf{H}}_i$, $\tilde{\mathbf{H}}_r$, and $\tilde{\mathbf{H}}_t$ on $x = 0$ plane are obtained from

$$\tilde{\mathbf{E}}_i = \hat{y}E_o e^{-jk_1 \sin\theta_1 z}, \quad \tilde{\mathbf{E}}_r = \hat{y}E_o\Gamma_\perp e^{-jk_1 \sin\theta_1 z}, \quad \tilde{\mathbf{E}}_t = \hat{y}E_o\tau_\perp e^{-jk_2 \sin\theta_2 z}.$$

as

$$\hat{z}\cdot\tilde{\mathbf{H}}_i = \frac{E_o \cos\theta_1}{\eta_1} e^{-jk_1 \sin\theta_1 z}, \quad \hat{z}\cdot\tilde{\mathbf{H}}_r = -\frac{E_o\Gamma_\perp \cos\theta_1}{\eta_1} e^{-jk_1 \sin\theta_1 z}, \quad \hat{z}\cdot\tilde{\mathbf{H}}_t = \frac{E_o\tau_\perp \cos\theta_2}{\eta_2} e^{-jk_2 \sin\theta_2 z}.$$

Clearly, given Snell's law, tangential continuity of the total field phasor $\tilde{\mathbf{H}}$ across $x = 0$ surface will then be satisfied for all z if and only if

$$\frac{\cos\theta_1}{\eta_1} - \frac{\cos\theta_1}{\eta_1}\Gamma_\perp = \frac{\cos\theta_2}{\eta_2}\tau_\perp.$$

Combining this with

$$1 + \Gamma_\perp = \tau_\perp,$$

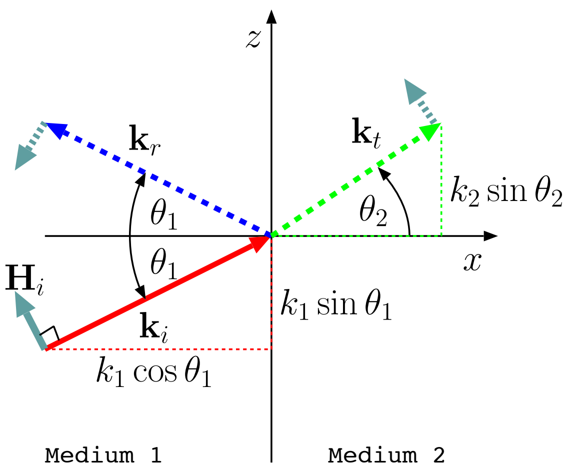
[1] Note that "Snell's law" can also be interpreted as having the components of wavevectors $\mathbf{k}_i$ and $\mathbf{k}_t$ equal along the interface between media 1 and 2.

we find that

$$\frac{\cos\theta_1}{\eta_1} - \frac{\cos\theta_1}{\eta_1}\Gamma_\perp = \frac{\cos\theta_2}{\eta_2}(1+\Gamma_\perp) \quad\Rightarrow\quad \Gamma_\perp = \frac{\eta_2\cos\theta_1 - \eta_1\cos\theta_2}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2}$$

and

$$\tau_\perp = 1 + \Gamma_\perp = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2}.$$



- **Conclusion:** In TE reflection problem, the Fresnel reflection and transmission coefficients are

$$\Gamma_\perp \equiv \frac{E_{yr}}{E_{yi}} = \frac{\eta_2\cos\theta_1 - \eta_1\cos\theta_2}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2} \text{ and } \tau_\perp = \frac{E_{yt}}{E_{yi}} = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2},$$

respectively. The coefficients enable us to express the reflected and transmitted wave phasors in terms of the incident-wave electric field phasor at the origin (i.e., E_{yi}).

Example 1: Medium 2 is vacuum while medium 1 has $\mu_1 = \mu_o$ and $\epsilon_1 = 2\epsilon_o$. Given that

$$\tilde{\mathbf{E}}_i = \hat{y} 5 e^{-jk_1(\cos 30^\circ x + \sin 30^\circ z)} \,\frac{\mathrm{V}}{\mathrm{m}},$$

determine $\tilde{\mathbf{E}}_r$, $\tilde{\mathbf{E}}_t$, and $\tilde{\mathbf{H}}_t$.

Solution: We have

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \sqrt{\frac{\mu_o}{2\epsilon_o}} = \frac{\eta_o}{\sqrt{2}} \text{ and } \eta_2 = \eta_o.$$

Also, according to Snell's law,

$$k_1 \sin\theta_1 = k_2 \sin\theta_2 \quad \Rightarrow \quad \sin\theta_2 = \frac{k_1}{k_2}\sin\theta_1 = \frac{\sqrt{\epsilon_1\mu_1}}{\sqrt{\epsilon_2\mu_2}}\sin\theta_1 = \sqrt{2}\sin 30^\circ = \frac{1}{\sqrt{2}},$$

indicating that

$$\theta_2 = 45^\circ.$$

Now, the reflection coefficient is

$$\Gamma_\perp \equiv \frac{E_{yr}}{E_{yi}} = \frac{\eta_2\cos\theta_1 - \eta_1\cos\theta_2}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2} = \frac{\eta_o\frac{\sqrt{3}}{2} - \frac{\eta_o}{\sqrt{2}}\frac{1}{\sqrt{2}}}{\eta_o\frac{\sqrt{3}}{2} + \frac{\eta_o}{\sqrt{2}}\frac{1}{\sqrt{2}}} = \frac{\frac{\sqrt{3}}{2} - \frac{1}{2}}{\frac{\sqrt{3}}{2} + \frac{1}{2}} \approx 0.268.$$

The transmission coefficient is

$$\tau_\perp = \frac{E_{yt}}{E_{yi}} = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2} = \frac{2\eta_o\frac{\sqrt{3}}{2}}{\eta_o\frac{\sqrt{3}}{2} + \frac{\eta_o}{\sqrt{2}}\frac{1}{\sqrt{2}}} \approx 1.268.$$

Consequently, the reflected and transmitted wave phasors are

$$\tilde{\mathbf{E}}_r = \hat{y}(5 \times 0.268) e^{-jk_1(-\cos 30^\circ x + \sin 30^\circ z)} \,\frac{\mathrm{V}}{\mathrm{m}}$$

and

$$\tilde{\mathbf{E}}_t = \hat{y}(5 \times 1.268)e^{-jk_2(\cos 45^\circ x + \sin 45^\circ z)} \ \frac{\text{V}}{\text{m}}.$$

Finally,

$$\begin{aligned}
\tilde{\mathbf{H}}_t &= \frac{\mathbf{k}_2 \times \tilde{\mathbf{E}}_t}{k_2 \eta_2} = \frac{(\cos 45^\circ \hat{x} + \sin 45^\circ \hat{z}) \times \hat{y}(5 \times 1.268)e^{-jk_2(\cos 45^\circ x + \sin 45^\circ z)}}{\eta_o} \\
&= \frac{(\hat{z} - \hat{x})(5 \times 1.268)e^{-jk_2(\cos 45^\circ x + \sin 45^\circ z)}}{120\pi\sqrt{2}} \ \frac{\text{A}}{\text{m}}.
\end{aligned}$$