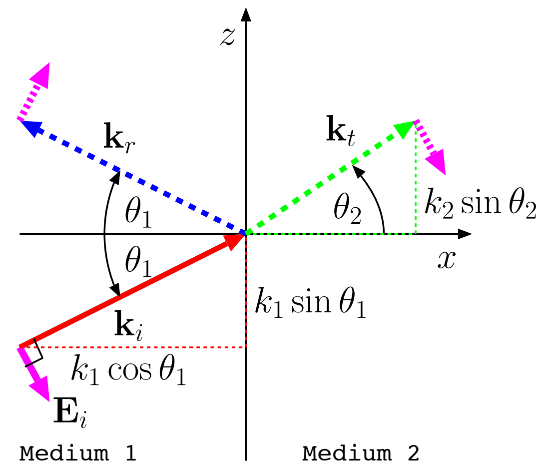
17 Reflection and transmission, TM mode



- In TM mode reflection problem, the plane-wave field phasors incident on the interface between medium 1 and 2 — $x = 0$ plane — are specified as

$$\tilde{\mathbf{H}}_i = \hat{y} H_o e^{-j\mathbf{k}_i \cdot \mathbf{r}} \text{ and } \tilde{\mathbf{E}}_i = -\eta_1 \frac{\mathbf{k}_i \times \tilde{\mathbf{H}}_i}{k_1},$$

where

$$\mathbf{k}_i = k_1(\hat{x}\cos\theta_1 + \hat{z}\sin\theta_1),$$

and

$$k_1 = \frac{\omega}{v_1}, \;\; v_1 = \frac{1}{\sqrt{\mu_1 \epsilon_1}}, \;\; \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}.$$

- In medium 1 we postulate a **reflected plane-wave** with field phasors

$$\tilde{\mathbf{H}}_r = \hat{y} H_o R e^{-j\mathbf{k}_r \cdot \mathbf{r}} \text{ and } \tilde{\mathbf{E}}_r = -\eta_1 \frac{\mathbf{k}_r \times \tilde{\mathbf{H}}_r}{k_1},$$

where

$$\mathbf{k}_r = k_1(-\hat{x}\cos\theta_1 + \hat{z}\sin\theta_1).$$

- In medium 2 we postulate a **transmitted plane-wave** with field phasors

$$\tilde{H}_t = \hat{y} H_o T e^{-j\mathbf{k}_t \cdot \mathbf{r}} \text{ and } \tilde{\mathbf{E}}_t = -\eta_2 \frac{\mathbf{k}_t \times \tilde{\mathbf{H}}_t}{k_2},$$

where

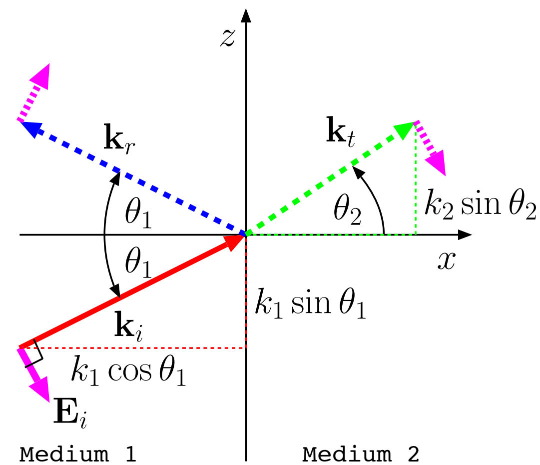
$$\mathbf{k}_t = k_2(\hat{x}\cos\theta_2 + \hat{z}\sin\theta_2).$$

and

$$k_2 = \frac{\omega}{v_2},\ \ v_2 = \frac{1}{\sqrt{\mu_2\epsilon_2}},\ \ \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}.$$

- To justify our postulates and determine a set of reflection and transmission coefficients R and T — defined in terms of magnetic field components — we will next apply the boundary conditions on $x = 0$ surface, where (using $x = 0$)

$$\tilde{\mathbf{H}}_i = \hat{y}H_o e^{-jk_1\sin\theta_1 z},\ \ \tilde{\mathbf{H}}_r = \hat{y}H_o R e^{-jk_1\sin\theta_1 z},\ \ \tilde{\mathbf{H}}_t = \hat{y}H_o T e^{-jk_2\sin\theta_2}.$$



Clearly, with these field components tangential continuity of the total field phasor $\tilde{\mathbf{H}}$ across $x = 0$ surface will be satisfied for all z if and only if

$$e^{-jk_1\sin\theta_1 z} + Re^{-jk_1\sin\theta_1 z} = Te^{-jk_2\sin\theta_2}$$

which — given Snell's law — is only possible (non-trivially) if

$$1 + R = T.$$

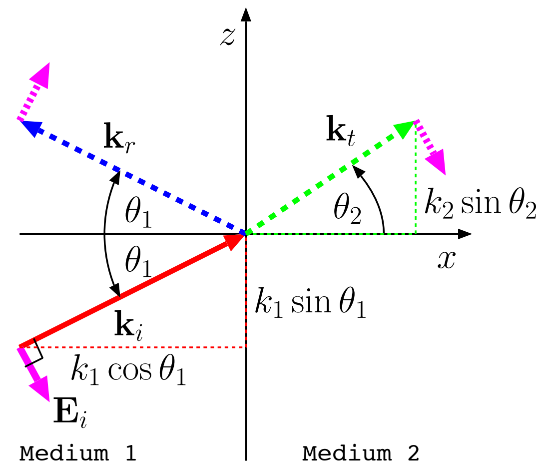
- Tangential components of $\tilde{\mathbf{E}}_i$, $\tilde{\mathbf{E}}_r$, and $\tilde{\mathbf{E}}_t$ on $x = 0$ plane are then obtained from

$$\tilde{\mathbf{H}}_i = \hat{y}H_o e^{-jk_1\sin\theta_1 z},\ \ \tilde{\mathbf{H}}_r = \hat{y}H_o R e^{-jk_1\sin\theta_1 z},\ \ \tilde{\mathbf{H}}_t = \hat{y}H_o T e^{-jk_2\sin\theta_2 z}.$$

as

$$\hat{z}\cdot\tilde{\mathbf{E}}_i = -\eta_1\cos\theta_1 H_o e^{-jk_1\sin\theta_1 z},\ \ \hat{z}\cdot\tilde{\mathbf{E}}_r = \eta_1\cos\theta_1 H_o R e^{-jk_1\sin\theta_1 z},\ \ \hat{z}\cdot\tilde{\mathbf{E}}_t = -\eta_2\cos\theta_2 H_o T e^{-jk_2\sin\theta_2 z}.$$

Clearly, given Snell's law, tangential continuity of the total field phasor $\tilde{\mathbf{E}}$ across $x = 0$ surface will be satisfied for all z if and only if



$$\eta_1 \cos\theta_1 (1 - R) = \eta_2 \cos\theta_2 T$$

Combining this with

$$1 + R = T,$$

we find that

$$\eta_1 \cos\theta_1 (1 - R) = \eta_2 \cos\theta_2 (1 + R) \quad \Rightarrow$$

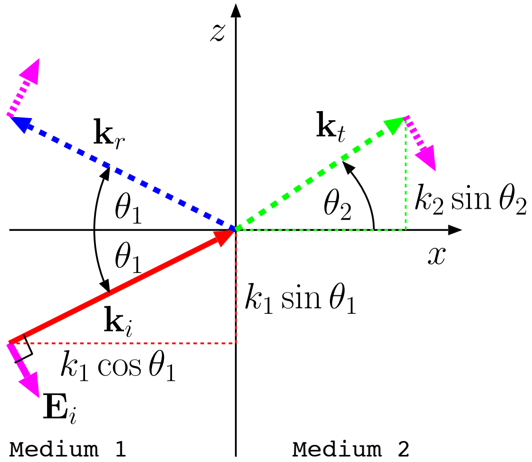
$$R = \frac{H_{yr}}{H_{yi}} = \frac{\eta_1 \cos\theta_1 - \eta_2 \cos\theta_2}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2} = \frac{E_r}{E_i} \equiv -\Gamma_\parallel = -\frac{E_{zr}}{E_{zi}},$$

and

$$1 + \frac{\eta_1 \cos\theta_1 - \eta_2 \cos\theta_2}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2} = T \quad \Rightarrow$$

$$T = \frac{H_{yt}}{H_{yi}} = \frac{2\eta_1 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2} = \frac{\eta_1}{\eta_2}\frac{E_t}{E_i} \equiv \frac{\eta_1}{\eta_2}\tau_\parallel.$$

Above, E_i, E_r, E_t refer to the amplitudes of the incident, reflected, and transmitted field vectors at the origin pointing in reference directions along $-\mathbf{k} \times \tilde{\mathbf{H}}$ indicated by the arrows shown in magenta in the margin.



- **Conclusion:** In TM case, the Fresnel reflection and transmission coefficients for plane-wave electric fields are

$$\Gamma_\parallel \equiv -\frac{E_r}{E_i} = \frac{E_{rz}}{E_{iz}} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1} \quad \text{and} \quad \tau_\parallel \equiv \frac{E_t}{E_i} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1},$$

  respectively. The coefficients enable us to express the reflected and transmitted wave phasors in terms of the incident-wave electric field phasor at the origin (i.e., E_i).

- Note that for $\theta_1 \to 0$, the Snell's law implies $\theta_1 \to 0$ also, in which case

$$\Gamma_\parallel = \frac{E_{zr}}{E_{zi}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \Gamma_\perp = \frac{E_{yr}}{E_{yi}}$$

  and

$$\tau_\parallel = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_2 + \eta_1} = \tau_\perp = \frac{E_{yt}}{E_{yi}}$$

  (as one would have hoped for) since the distinction between TE and TM cases vanishes in this limit.

Example 1: Medium 2 is vacuum while medium 1 has $\mu_1 = \mu_o$ and $\epsilon_1 = 2\epsilon_o$. A TM mode plane-wave with an electric field amplitude of 1 V/m is incident on medium 2 with an angle of incidence of $\theta_1 = 30°$. Determine the wave-field phasors $\tilde{\mathbf{E}}_i$, $\tilde{\mathbf{E}}_r$, and $\tilde{\mathbf{E}}_t$.

Solution: The described incident wave field can be represented as

$$\tilde{\mathbf{E}}_i = (\sin 30°\hat{x} - \cos 30°\hat{z})e^{-jk_1(\cos 30° x + \sin 30° z)}\,\frac{\text{V}}{\text{m}}.$$

Also

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \sqrt{\frac{\mu_o}{2\epsilon_o}} = \frac{\eta_o}{\sqrt{2}} \text{ and } \eta_2 = \eta_o.$$

Snell's law gives

$$k_1 \sin\theta_1 = k_2 \sin\theta_2 \;\Rightarrow\; \sin\theta_2 = \frac{k_1}{k_2}\sin\theta_1 = \frac{\sqrt{\epsilon_1\mu_1}}{\sqrt{\epsilon_2\mu_2}}\sin\theta_1 = \sqrt{2}\sin 30° = \frac{1}{\sqrt{2}},$$

indicating that

$$\theta_2 = 45°.$$

The TM mode reflection coefficient is

$$\Gamma_\parallel \equiv -\frac{E_r}{E_i} = \frac{\eta_2\cos\theta_2 - \eta_1\cos\theta_1}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1} = \frac{\eta_o\frac{1}{\sqrt{2}} - \frac{\eta_o}{\sqrt{2}}\frac{\sqrt{3}}{2}}{\eta_o\frac{1}{\sqrt{2}} + \frac{\eta_o}{\sqrt{2}}\frac{\sqrt{3}}{2}} = \frac{1-\sqrt{3}/2}{1+\sqrt{3}/2} \approx 0.0718.$$

The transmission coefficient is

$$\tau_\parallel \equiv \frac{E_t}{E_i} = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_2 + \eta_1\cos\theta_1} = \frac{2\eta_o\frac{\sqrt{3}}{2}}{\eta_o\frac{1}{\sqrt{2}} + \frac{\eta_o}{\sqrt{2}}\frac{\sqrt{3}}{2}} \approx 1.3127$$

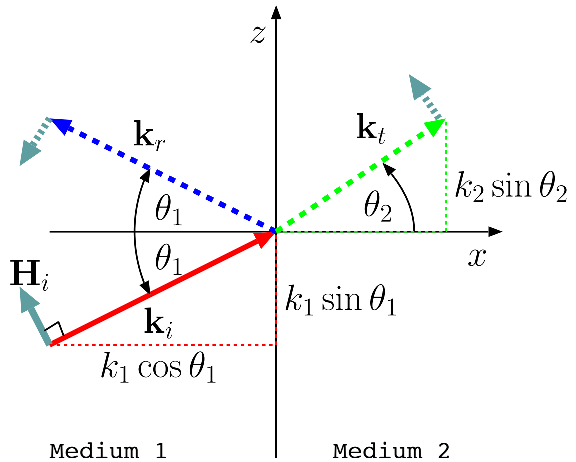
Consequently, the reflected and transmitted wave phasors are

$$\tilde{\mathbf{E}}_r = -0.0718(\sin 30°\hat{x} + \cos 30°\hat{z})e^{-jk_1(-\cos 30° x + \sin 30° z)}\,\frac{\text{V}}{\text{m}}.$$

and

$$\tilde{\mathbf{E}}_t = 1.3127(\sin 45^\circ \hat{x} - \cos 45^\circ \hat{z}) e^{-jk_2(\cos 45^\circ x + \sin 45^\circ z)} \frac{\text{V}}{\text{m}}.$$

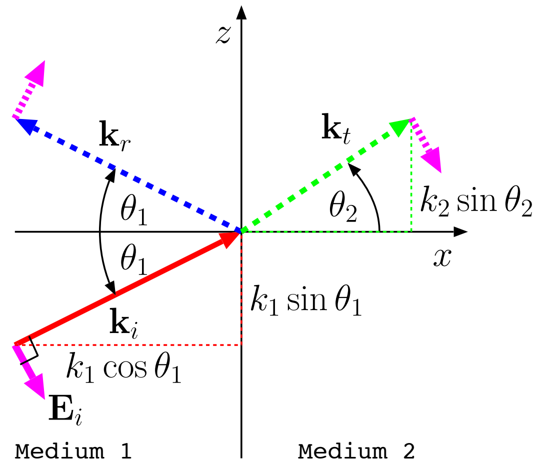
TE reflection:



$$\Gamma_\perp \equiv \frac{E_{yr}}{E_{yi}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

$$\tau_\perp \equiv \frac{E_{yt}}{E_{yi}} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2},$$

TM reflection:



$$\Gamma_\parallel \equiv -\frac{E_r}{E_i} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1} = \frac{E_{zr}}{E_{zi}}$$

$$\tau_\parallel \equiv \frac{E_t}{E_i} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}.$$

Brewster's angle:

For *diamagnetic* and *paramagnetic* materials which cover a vast amounts of media of interest in EM and optical applications, we have $\mu \approx \mu_o$. For TE and TM reflection problems between diamagnetic and/or paramagnetic materials it is therefore possible to take $\mu_2 = \mu_1$ and simplify the reflection and transmission coefficient formulae above.

- For the case $\mu_2 = \mu_1$, $\Gamma_\perp = 0$ iff $\eta_2 = \eta_1$, i.e, $\epsilon_2 = \epsilon_1$, but it is possible to have $\Gamma_\parallel = 0$ with $\eta_2 \neq \eta_1$ at a special angle θ_1 known as **Brewster's angle**, θ_p, examined in this section.

TE reflection:

$$\Gamma_\perp \equiv \frac{E_{yr}}{E_{yi}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

$$\tau_\perp \equiv \frac{E_{yt}}{E_{yi}} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2},$$

TM reflection:

$$\Gamma_\parallel \equiv -\frac{E_r}{E_i} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1} = \frac{E_{zr}}{E_{zi}}$$

$$\tau_\parallel \equiv \frac{E_t}{E_i} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}.$$

- Note that in view of Snell's law

$$\sqrt{\mu_2\epsilon_2}\sin\theta_2 = \sqrt{\mu_1\epsilon_1}\sin\theta_1$$

  we have

$$\Gamma_\parallel = -\frac{E_r}{E_i} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1} = 0$$

only when

$$\eta_1 \cos\theta_1 = \eta_2 \cos\theta_2 = \eta_2\sqrt{1-\sin^2\theta_2} = \eta_2\sqrt{1-\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_1}.$$

Squaring this we get

$$\frac{\mu_1}{\epsilon_1}(1-\sin^2\theta_1) = \frac{\mu_2}{\epsilon_2}(1-\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_1) \quad\Rightarrow\quad \sin^2\theta_1 = \frac{1-\frac{\mu_2}{\mu_1}\frac{\epsilon_1}{\epsilon_2}}{(1-\frac{\epsilon_1}{\epsilon_2})(1+\frac{\epsilon_1}{\epsilon_2})}.$$

- For $\mu_2 = \mu_1$, this yields

$$\theta_1 = \sin^{-1}\sqrt{\frac{1}{1+\frac{\epsilon_1}{\epsilon_2}}} \equiv \theta_p$$

which simplifies as the **Brewster's angle**

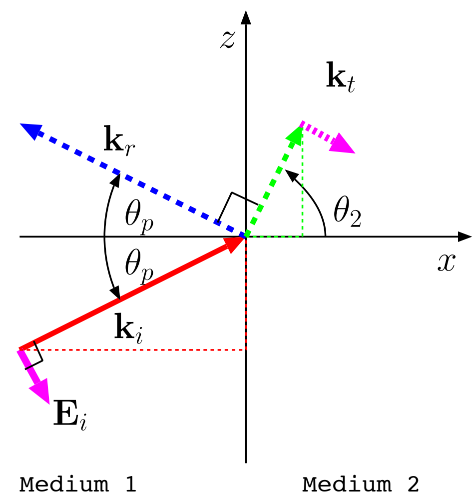
$$\theta_p = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_2+\epsilon_1}} \quad\Rightarrow\quad \theta_p = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}.$$

- A physical insight to Brewster's angle θ_p can be gained by noting that (as verified below)

$$\theta_p + \theta_2 = 90^\circ,$$

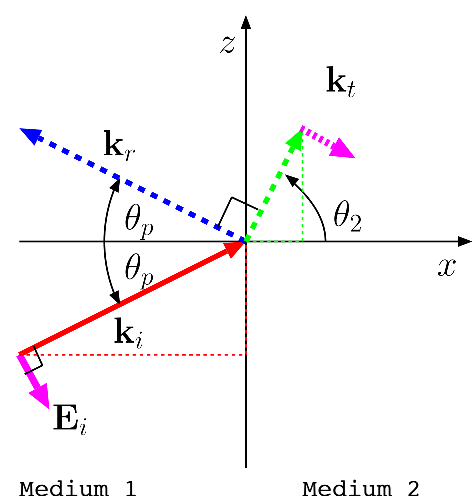
implying that vector $-\tilde{\mathbf{E}}_t$ in medium 2 is co-aligned with wavevector $\mathbf{k}_r$ of the zero-amplitude reflected wave in medium 1 as shown in the margin.

Geometry at Brewster's angle



$\epsilon_1 > \epsilon_2 \quad\Leftrightarrow\quad \theta_2 > \theta_1$

Geometry at Brewster's angle



$\epsilon_1 > \epsilon_2 \quad\Leftrightarrow\quad \theta_2 > \theta_1$

– Now, the physical cause of the plane-wave $\tilde{\mathbf{E}}_r$ is the superposition of dipole radiations of polarized molecules in medium 2 into medium 1, behaving like a giant 3D antenna array.

– However propagation direction $\mathbf{k}_r$ of wave $\tilde{\mathbf{E}}_r$ is the dipole axis of these molecules when $\theta_1 = \theta_p$, in which case radiation amplitude becomes zero because dipoles do not radiate along their axes as we have seen earlier on (they radiate best in the broadside direction)!

Verification of $\theta_p + \theta_2 = 90°$: For $\mu_2 = \mu_1$, Snell's law simplifies as

$$\sqrt{\epsilon_2}\sin\theta_2 = \sqrt{\epsilon_1}\sin\theta_1,$$

yielding

$$\sin\theta_2 = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_1.$$

For $\theta_1 = \theta_p$, we have

$$\sin\theta_2 = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_p = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sqrt{\frac{\epsilon_2}{\epsilon_2+\epsilon_1}} = \sqrt{\frac{\epsilon_1}{\epsilon_2+\epsilon_1}},$$

implying that

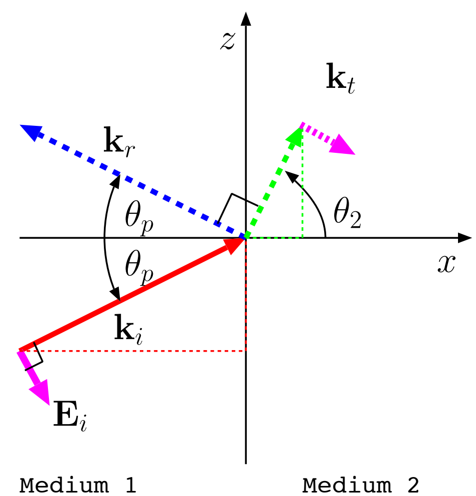
$$\sin^2\theta_2 = \frac{\epsilon_1}{\epsilon_2+\epsilon_1} = \frac{\epsilon_2+\epsilon_1-\epsilon_2}{\epsilon_2+\epsilon_1} = 1 - \frac{\epsilon_2}{\epsilon_2+\epsilon_1} = 1 - \sin^2\theta_p = \cos^2\theta_p.$$

Thus

$$\sin\theta_2 = \cos\theta_p,$$

a telltale sign that θ_p and the corresponding θ_2 add up to 90°! (QED)

Geometry at Brewster's angle



$\epsilon_1 > \epsilon_2 \quad \Leftrightarrow \quad \theta_2 > \theta_1$

- Finally, there is no Brewster's angle for TE mode reflections because in the TE case $\mathbf{k}_r$ is unconditionally in the broadside direction of $\tilde{\mathbf{E}}_r$ polarized dipoles (in $\hat{y}$ direction). It is easy to see that for $\mu_2 = \mu_1$, $\Gamma_\perp = 0$ iff $\epsilon_2 = \epsilon_1$:

  Verification: According to Snell's law

$$\sqrt{\epsilon_2} \sin\theta_2 = \sqrt{\epsilon_1} \sin\theta_1$$

  while

$$\Gamma_\perp = \frac{E_{yr}}{E_{yi}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2} = 0$$

  only when

$$\eta_1 \cos\theta_2 = \eta_2 \cos\theta_1.$$

  Dividing Snell's law with this relationship we get

$$\frac{\sqrt{\epsilon_2}}{\sqrt{\mu_1/\epsilon_1}} \tan\theta_2 = \frac{\sqrt{\epsilon_1}}{\sqrt{\mu_1/\epsilon_2}} \tan\theta_1 \quad \Rightarrow \quad \tan\theta_2 = \tan\theta_1.$$

  But this condition of $\theta_2 = \theta_1$ is only permitted by Snell's law if $\eta_2 = \eta_1$, the trivial case of no practical interest.

- Read pp 322-323 in Rao for a discussion of the applications of Brewster's angle.

- One simple application: reflected light from ground is typically TE polarized (parallel to the ground) because the TM component of light reflects poorly because typically θ_1 may be close to θ_p. It is easy to eliminate TE polarized glare from the ground by using polarized eyeglasses which only transmit the TM component of light (polarized vertically). Note that this application also explains why the Brewster's angle θ_p is also known as "polarizing angle".