19 Total internal reflection (TIR) and evanescent waves

- Consider a TE- or TM-polarized wave (or a superposition) incident on an interface at $x = 0$ surface as depicted in the margin at an incidence angle θ_1.

- Independent of the polarization of the incident wave, the angle of transmitted wave θ_2 can be found using Snell's law

$$k_1 \sin\theta_1 = k_2 \sin\theta_2 \quad \Rightarrow \quad \sqrt{\mu_{1r}\epsilon_{1r}} \sin\theta_1 = \sqrt{\mu_{2r}\epsilon_{2r}} \sin\theta_2$$

assuming lossless media on either side of the interface, where

$$\mu_r \equiv \frac{\mu}{\mu_o} \text{ and } \epsilon_r \equiv \frac{\epsilon}{\epsilon_o}$$

are the relative permeability and permittivity, respectively, of the propagation media. Moreover,

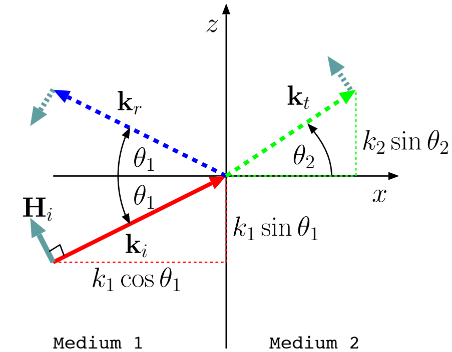
$$\sqrt{\mu_r \epsilon_r} = \frac{\sqrt{\mu\epsilon}}{\sqrt{\mu_o \epsilon_o}} = \frac{c}{v_p} \equiv n$$

above can be referred to as the **refractive index** of the propagation medium.

- Snell's law, expressed in terms of refractive index,

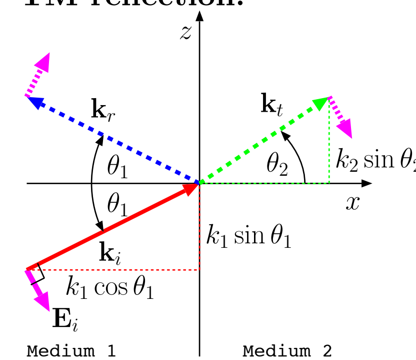
$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad \Rightarrow \quad \sin\theta_2 = \frac{n_1}{n_2} \sin\theta_1$$

TE reflection:



$$\frac{E_{yr}}{E_{yi}} = \frac{\eta_2 \cos\theta_1 - \eta_1 \cos\theta_2}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$
$$\frac{E_{yt}}{E_{yi}} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_1 + \eta_1 \cos\theta_2}$$

TM reflection:



$$-\frac{E_r}{E_i} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$
$$\frac{E_t}{E_i} = \frac{2\eta_2 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}.$$

Refractive index:

$$n = \frac{c}{v_p} = \sqrt{\mu_r \epsilon_r}$$

shows that for a given θ_1, the corresponding $\sin\theta_2$ can be in excess of 1 when $n_1 > n_2$, that is, for propagation from a high refractive index (optically thick) material such as glass into a lower refractive index (optically thin) material such as air.

– ***For example***: if $\frac{n_1}{n_2} = 1.5$ and $\theta_1 = 45°$, then

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 = 1.5\sin 45° = \frac{1.5}{\sqrt{2}} \approx \frac{1.5}{1.41} > 1.$$

But, $\sin\theta_2$ **in excess of 1 cannot be solved for** θ_2 **as if it were a "regular" angle**[1] **describing the elevation of vector** $\mathbf{k}_t$ **above the** x**-axis.**

- In general when $n_1 > n_2$ and the incidence angle

$$\theta_1 > \sin^{-1}\frac{n_2}{n_1} = \sin^{-1}\sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}} \equiv \theta_c$$

we will have $\sin\theta_2$ in excess of 1 and $\cos\theta_2 = \sqrt{1-\sin^2\theta_2}$ purely imaginary.

Critical angle:

$\theta_c = \sin^{-1}\frac{n_2}{n_1}$

– in such situations use $\sin\theta_2$ and $\cos\theta_2 = \sqrt{1-\sin^2\theta_2}$ directly in the expressions for $\mathbf{k}_t$, Γ, and τ as illustrated below.

[1]Nor if $\sin\theta_2$ is complex valued because medium 2 is lossy and we need to use $\epsilon_{2r} = \frac{\epsilon_2}{\epsilon_o} + \frac{\sigma_2}{j\omega\epsilon_o}$ in Snell's law (as we already did in Lecture 18).

– as we will see the situation corresponds to having a **total internal reflection (TIR)** in medium 1 and establishing an **evanescent wave** (a special form of non-uniform plane wave with an imaginary valued k_{2x}) in medium 2.

Total internal reflection (TIR)

and

evanescent wave

– ***For example***: for $\frac{n_1}{n_2} = 1.5$ we have

$$\theta_c = \sin^{-1}\frac{n_2}{n_1} = \sin^{-1}\frac{1}{1.5} \approx 41.81°,$$

which is less than $\theta_1 = 45°$ which is why we find $\sin\theta_2 > 1$ in the above example (see margin for an example plot of this configuration in the context of a glass prism with $n = 1.5$).

- To understand the field topologies for $\theta_1 > \theta_c = \sin^{-1}\frac{n_2}{n_1}$ let us examine the reflected and transmitted field phasors for, say, the TE-polarization as θ_1 approaches and then exceeds θ_c. We will simplify this exercise by taking $\mu_1 = \mu_2 = \mu_o$ so that the refractive index

$$n_{1,2} = \sqrt{\epsilon_{r1,2}}$$

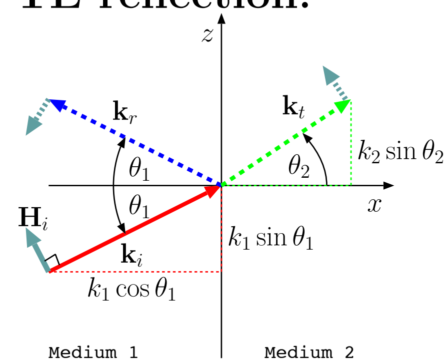
in Snell's law, and so that the reflection and transmission coefficients for the TE-mode shown in the margin can be expressed as

$$\Gamma_\perp = \frac{E_{yr}}{E_{yi}} = \frac{\eta_2\cos\theta_1 - \eta_1\cos\theta_2}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2} \quad\Rightarrow\quad \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

and

$$\tau_\perp = 1 + \Gamma_\perp = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}.$$

TE reflection:



$$\frac{E_{yr}}{E_{yi}} = \frac{\eta_2\cos\theta_1 - \eta_1\cos\theta_2}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2}$$
$$\frac{E_{yt}}{E_{yi}} = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2}$$

We have, using Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad \Rightarrow \quad \sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1,$$

and, therefore,

$$\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_1}$$

in the coefficients above.

- For

$$\theta_1 \geq \theta_c = \sin^{-1}\frac{n_2}{n_1} \quad \Leftrightarrow \quad \sin^2\theta_1 \geq \frac{n_2^2}{n_1^2},$$

we have a ***purely imaginary***

$$\cos\theta_2 = \sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_1} = \pm j\alpha, \ \text{ with } \ \alpha = \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1},$$

in which case

$$\Gamma_\perp = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2} = \frac{n_1\cos\theta_1 \mp jn_2\alpha}{n_1\cos\theta_1 \pm jn_2\alpha} = 1\angle \mp 2\tan^{-1}(\frac{n_2\alpha}{n_1\cos\theta}).$$

and

$$\tau_\perp = 1 + \Gamma_\perp = 1 + 1\angle \mp 2\tan^{-1}(\frac{n_2\alpha}{n_1\cos\theta}).$$

Note that

1. $|\Gamma_\perp| = |E_{yr}/E_{yi}| = 1$ **at all** $\theta_1 \geq \theta_c$**, a condition known as** *total internal reflection* **(TIR).**

2. $\tau_\perp = E_{yt}/E_{yi} \neq 0$ **in general and therefore a non-zero transmitted field exists in medium 2 despite TIR — this field in medium 2 has** *evanescent wave* **character described below.**

   - We can express the transmitted TE-mode field phasor for $\theta_1 \geq \theta_c$ as

$$\tilde{\mathbf{E}}_t = \hat{y}E_{yt}e^{-j\mathbf{k}_t\cdot\mathbf{r}} = \hat{y}E_{yt}e^{-jk_2(\cos\theta_2 x + \sin\theta_2 z)}$$

where

$$E_{yt} = E_{yi}\left(1 + \frac{n_1\cos\theta_1 \mp jn_2\alpha}{n_1\cos\theta_1 \pm jn_2\alpha}\right),$$

$$k_2\sin\theta_2 = k_1\sin\theta_1 \quad \text{(Snell's law)},$$

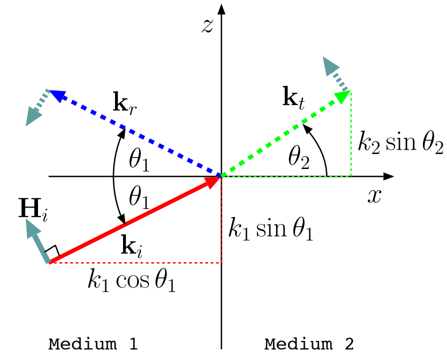
and

$$k_2\cos\theta_2 = k_2(\pm j\alpha) = \pm jk_2\alpha, \quad \text{where } \alpha = \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}.$$

Thus,

$$\begin{aligned}\tilde{\mathbf{E}}_t &= \hat{y}E_{yi}\left(1 + \frac{n_1\cos\theta_1 \mp jn_2\alpha}{n_1\cos\theta_1 \pm jn_2\alpha}\right) e^{-jk_1\sin\theta_1 z}\, e^{-j(\pm jk_2\alpha)x} \\ &= \hat{y}E_{yi}\left(1 + \frac{n_1\cos\theta_1 \mp jn_2\alpha}{n_1\cos\theta_1 \pm jn_2\alpha}\right) e^{-jk_1\sin\theta_1 z}\, e^{\pm k_2\alpha x}.\end{aligned}$$

TE reflection:



$$\frac{E_{yr}}{E_{yi}} = \frac{\eta_2\cos\theta_1 - \eta_1\cos\theta_2}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2}$$

$$\frac{E_{yt}}{E_{yi}} = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2}$$

- Depending on which root we select — + or − — we have two candidate solutions for medium 2 satisfying the plane-wave form of Maxwell's equations.

  – One of the solutions blows up as $x \to \infty$, which, therefore cannot be a physical solution, leaving us with

  $$\tilde{\mathbf{E}}_t = \hat{y} E_{yi}(1 + \Gamma_\perp) e^{-jk_1 \sin\theta_1 z} e^{-k_2 \alpha x}$$

  with

  $$\Gamma_\perp = \frac{n_1 \cos\theta_1 + jn_2\alpha}{n_1 \cos\theta_1 - jn_2\alpha} \text{ and } \alpha = \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}$$

  as our expression for the *evanescent* wave established during a total internal reflection event when

  $$\theta_1 \geq \theta_c = \sin^{-1}\frac{n_2}{n_1}.$$

  – This solution should fit (as we are about to see) the plane-wave form of Maxwell's equations with

  $$\mathbf{k} = \mathbf{k}_t = k_2(\hat{x}\cos\theta_2 + \hat{z}\sin\theta_2) = -jk_2\alpha\hat{x} + k_1\sin\theta_1\hat{z}$$

  which

  1. is perpendicular to $\tilde{\mathbf{E}}_t \propto \hat{y}$ as required — i.e., $\mathbf{k}_t \cdot \tilde{\mathbf{E}}_t = 0$ (Gauss's law),

2. implies a *complex valued* unit vector

$$\hat{k} = \frac{\mathbf{k}_t}{k_2} = -j\alpha\hat{x} + \frac{k_1}{k_2}\sin\theta_1\hat{z}$$

which satisfies

$$\hat{k}\cdot\hat{k} = -\alpha^2 + \frac{k_1^2}{k_2^2}\sin^2\theta_1 = -(\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1) + \frac{n_1^2}{n_2^2}\sin^2\theta_1 = 1$$

as required, and

3. implies a transmitted magnetic field intensity phasor

$$\tilde{\mathbf{H}}_t = \frac{\hat{k}\times\tilde{\mathbf{E}}_t}{\eta_2} = \frac{E_{yi}}{\eta_2}(1+\Gamma_\perp)e^{-jk_1\sin\theta_1 z}e^{-k_2\alpha x}(-j\alpha\hat{z} - \frac{k_1}{k_2}\sin\theta_1\hat{x})$$

which is of course transverse to $\hat{k}$ *ipso facto*.

– It remains to show that

$$\hat{x}\cdot\langle\mathbf{E}_t\times\mathbf{H}_t\rangle = 0$$

so that

$$\hat{x}\cdot\langle\mathbf{E}_i\times\mathbf{H}_i\rangle + \hat{x}\cdot\langle\mathbf{E}_r\times\mathbf{H}_r\rangle = 0$$

— demanded by TIR — is satisfied.

Verification:

$$\begin{aligned}\langle\mathbf{E}_t\times\mathbf{H}_t\rangle &= \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}}_t\times\tilde{\mathbf{H}}_t^*\} \\ &= \frac{|E_{yi}|^2|\tau_\perp|^2}{2\eta_2}e^{-2k_2\alpha x}\mathrm{Re}\{\hat{y}\times(j\alpha\hat{z} - \frac{k_1}{k_2}\sin\theta_1\hat{x})\} \\ &= \frac{|E_{yi}|^2|\tau_\perp|^2}{2\eta_2}e^{-2k_2\alpha x}\frac{k_1}{k_2}\sin\theta_1\hat{z}.\end{aligned}$$

Evidently, timed averaged power flux is directed along the interface and has no component along $\hat{x}$ normal to the reflecting interface.

Example 1: Consider a uniform plane wave propagating in quartz with $\epsilon_r = 2.25$ and $n = \sqrt{\epsilon_r} = 1.5$ incident on a quartz/air interface at an incidence angle of 45°. Determine the evanescent field phasor $\tilde{\mathbf{E}}_t$ established in air outside the quartz slab. Assume that the incident wave is TE polarized and the wavelength is 1 mm within the quartz.

Solution: With $\theta_1 = 45°$ and $n_1 = 1.5$, $n_2 = 1$, we have $\cos\theta_2 = -jk_2\alpha$ with

$$\alpha = \sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1} = \sqrt{\frac{2.25}{1}\sin^2 45° - 1} = \sqrt{\frac{2.25}{2} - 1} = \sqrt{\frac{9}{8} - 1} = \frac{1}{2\sqrt{2}}.$$

Hence,

$$\Gamma_\perp = \frac{n_1\cos\theta_1 + jn_2\alpha}{n_1\cos\theta_1 - jn_2\alpha} = \frac{\frac{3}{2}\frac{1}{\sqrt{2}} + j1\frac{1}{2\sqrt{2}}}{\frac{3}{2}\frac{1}{\sqrt{2}} + -j1\frac{1}{2\sqrt{2}}} = \frac{3+j1}{3-j1} = \frac{8+j6}{10} = 0.8 + j0.6$$

and

$$\tau_\perp = 1 + \Gamma_\perp = 1.8 + j0.6.$$

Also, since $\lambda_1 = 1$ mm, it follows that

$$k_1 = \frac{2\pi}{\lambda_1} = 2\pi\frac{\text{rad}}{\text{mm}} = k_o n_1 = k_o 1.5, \quad k_o = \frac{k_1}{1.5} = \frac{4\pi}{3}\frac{\text{rad}}{\text{mm}},$$

where $k_o \equiv \omega/c$ is the free space wavenumber also applicable in air (i.e., k_2). Therefore, we have

$$\tilde{\mathbf{E}}_t = \hat{y}E_{yi}(1+\Gamma_\perp)e^{-jk_1\sin\theta_1 z}e^{-k_2\alpha x}$$

$$\hat{y}E_{yi}(1.8 + j0.6)e^{-j\frac{2\pi}{\sqrt{2}}z}e^{-\frac{2\pi}{3\sqrt{2}}x}\,\frac{\text{V}}{\text{m}}$$

where x and z are used in mm units. This is an example of an evanescent field that "hugs" the the quartz/air interface on the air side.

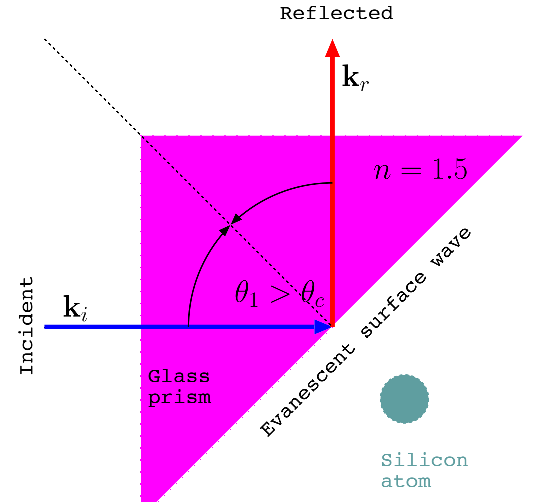
- **To summarize:** TIR that occurs when $\theta_1 \geq \theta_c$ is accompanied by an evanescent transmitted wave.

- **The evanescent wave:**

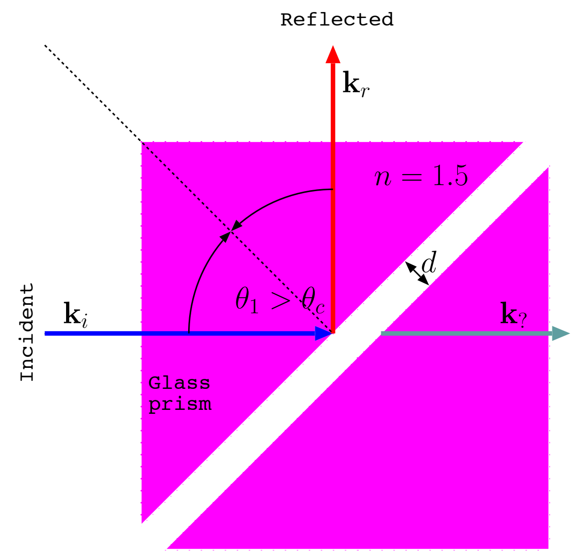
An evanescent wave is a *non-uniform TEM wave* since the field vector in non-uniform on surfaces of constant phase.

  1. has a decaying amplitude with distance away from the reflecting interface (that is along x) and carries no average power away from the interface,
  2. it exhibits a phase variation along the interface (that is along z) that matches the phase variations of the incident and reflected waves in medium 1,
  3. it carries average power only along the interface and only close to the interface because of the $e^{-2k_2\alpha x}$ factor — thus it is also known as a **surface wave**,
  4. it can be perturbed by introducing some new materials into region 2 to start drawing energy towards medium 2 — this is the topic of frustrated TIR to be examined next.

Frustrated TIR and tunneling:



- Suppose a silicon atom is brought to a location right next the prism as shown in the margin where an evanescent wave is present.
  - What happens then to the evanescent and total internal reflected waves to either sides of the diagonal face of the prism?
  - What happens when the atom is replaced by another prism placed, as shown in the second diagram in the margin, at a distance d away from the diagonal face?
- In the first instance, the silicon atom will be stretched into a polarized dipole by the action of the time-varying evanescent electric field outside the prism and therefore it will radiate like an oscillating Hertzian dipole antenna at the frequency of the evanescent wave.



- The radiation field of the atomic dipole will then superpose on the evanescent and internally reflected fields, modifying them both, and enabling the extraction of power from the incident wave to be transported away from the TIR interface.
  - This is an elementary example of what is known as energy "tunneling".
- The tunneling phenomenon becomes more pronounced when the atom is replaced with a second prism (a whole array of silicon atoms mixed

with oxygen atoms) as illustrated in the margin.

- The described phenomenon is known as “frustrated” TIR, because the presence of the second prism will perturb the reflected wave substantially when the gap width d between the prisms is a small fraction of a wavelength λ.
- The double prism arrangement shown in the margin can be used as a “practical” beam splitter at optical frequencies by adjusting d/λ.
- A quantitative treatment of the tunneling problem will be presented in Lecture 24 in a multiple slab geometry involving evanescent regions.