20 Doppler shift and Doppler radars

- **Doppler radars** make a use of the **Doppler shift** phenomenon to detect the motion of EM wave reflectors of interest — e.g., a police Doppler radar aims to identify the speed of a vehicle in relative motion.
  - In this lecture we will describe the general principle of how a Doppler radar works and also learn about the Doppler shift phenomenon in non-relativistic and relativistic limits.

Doppler radar:

- Consider a stationary dipole located at the origin excited by a co-sinusoidal input current

$$i(t) = I_o \cos(\omega t) \propto e^{j\omega t} + e^{-j\omega t} \equiv e^{j\omega t} + \text{cc}$$

where "cc" refers to the complex conjugate of the term preceding it.

- The dipole will radiate a spherical wave field

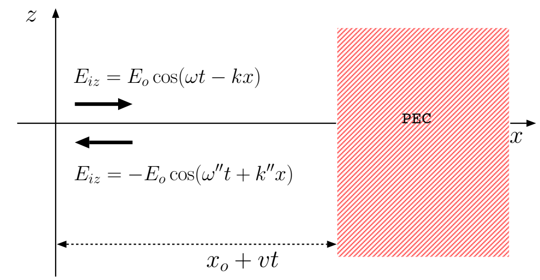
$$\mathbf{E}(\mathbf{r}, t) \propto \cos(\omega t - kr) \propto e^{j(\omega t - kr)} + \text{cc}$$

where

$$\frac{\omega}{k} = c$$

assuming propagation in vacuum or air.

- Consider now a car speeding away with velocity v from the dipole along the x-axis having an instantaneous location



$$x(t) = x_o + vt$$

at time t. The field at the location of the car at time t will then be

$$\propto \cos(\omega t - k(x_o + vt)) = \cos((\omega - kv)t - kx_o) \propto e^{j((\omega - kv)t - kx_o)} + \text{cc}.$$

- An induced surface current $\propto \cos(\omega' t - kx_o)$ on the car's body oscillating at a frequency

$$\omega' = \omega - kv$$

will then radiate like a collection of dipoles, producing a "reflected field"

$$\propto \cos(\omega' t - kx_o - k(x_o + vt)) = \cos((\omega - 2kv)t - 2kx_o) \propto e^{j((\omega - 2kv)t - 2kx_o)} + \text{cc}$$

detected back at the location of the original dipole — in this waveform we have included an additional phase delay of $k(x_o + vt)$ to account for the return trip of the reflected wave. Clearly, the reflected field oscillates with the frequency

$$\omega'' = \omega' - kv = \omega - 2kv$$

in the reference frame of the stationary dipole.

  - If the dipole is arranged to detect the reflected wave field (using a T/R switch — a radar jargon implying that the antenna is switched to connect to the input port of a receiving device shortly after the transmission of a burst of EM wave), then the velocity of the car, v, can be obtained from "two-way" Doppler shifted frequency ω''. That's how police radars work.

Note that

$$\omega' = \omega - kv$$

– positive v (motion away from the radar antenna) causes $\omega'' < \omega$ and is referred to as **redshift**, whereas

$$\omega'' = \omega - 2kv$$

– negative v (motion toward the radar antenna) causes $\omega'' > \omega$ and is referred to as **blueshift**.

Doppler shift in relativistic and non-relativistic limits:

- The "one-way" and "two-way" Doppler shift formulae

$$\omega' = \omega - kv \quad \text{and} \quad \omega'' = \omega - 2kv$$

obtained above, where v is the relative[1] radial recession velocity of the radiator and the observer, are valid only when $|v| \ll c$.

The reason for this is, our analysis above, leading to these formulae, neglected an important detail that according to Maxwell's equations we need to have

$$\text{not only } \frac{\omega}{k} = c, \quad \text{but also } \frac{\omega'}{k'} = c,$$

[1]It does not matter whether the radiator or the observer is "moving" since motion is always *relative*.

whereas we have, in effect, used an inconsistent relation $\frac{\omega'}{k} = c$ at an intermediate stage.

- This inconsistency produces a negligible error if $|v| \ll c$ (the usual case pertinent for police radar applications) but the errors are unacceptably large if $|v|$ approaches c (like in Fermilab).

• We will refer to the approximate Doppler shift formulae given above as ***non-relativistic*** **Doppler formulae** — they are to be used if and only if $|v/c| \ll 1$, i.e., in the non-relativistic limit.

• ***Relativistic*** **Doppler formulae** that can be used unconditionally (and most importantly for $|v/c|$ approaching unity) are

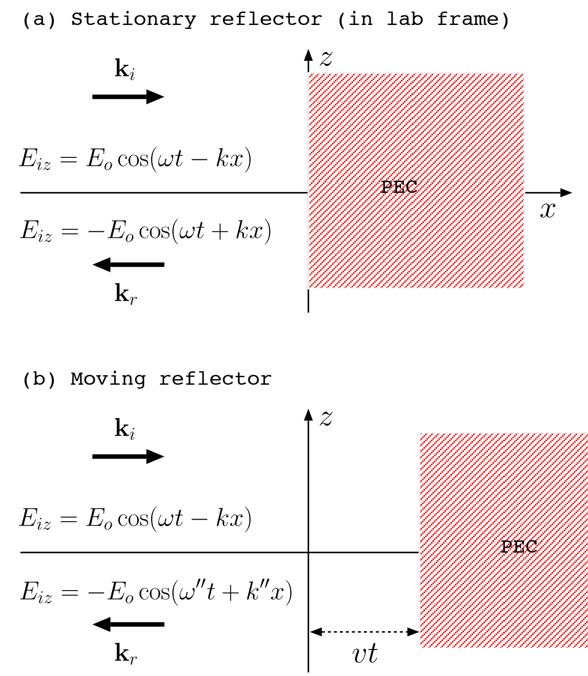
$$\omega' = \omega\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \quad \text{and} \quad \omega'' = \omega'\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = \omega\frac{1-\frac{v}{c}}{1+\frac{v}{c}}.$$

- Before deriving these relativistic formulae (correct for all v), let us note that they reduce to the non-relativistic formula if $|v/c| \ll 1$. In that case we have, for instance,

$$\begin{aligned}\omega' &= \omega\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = \omega\frac{(1-\frac{v}{c})^{1/2}}{(1+\frac{v}{c})^{1/2}} = \omega(1-\frac{v}{c})^{1/2}(1+\frac{v}{c})^{-1/2} \\ &\approx \omega(1-\frac{v}{2c})(1-\frac{v}{2c}) \approx \omega(1-\frac{v}{c}) = \omega - kv.\end{aligned}$$

Derivation of the relativistic formula:

To derive the relativistic Doppler shift formulae we will not need complicated relativistic transformation formulae discussed in PHYS 325 (also summarized in ECE 329 notes). It is sufficient that we make a careful use of Maxwell's equations in developing an accurate model of a field reflected from a reflector in motion as shown next:



- Consider a plane TEM wave in free-space,

$$\mathbf{E}_i(x,t) = \hat{z} E_o \cos(\omega t - kx),$$

incident on a conducting surface at $x = 0$ plane from the left such that

$$k = \frac{\omega}{c}.$$

The wave will be reflected to produce

$$\mathbf{E}_r(x,t) = -\hat{z} E_o \cos(\omega t + kx)$$

so that the total tangential field at $x = 0$ plane

$$\hat{z} \cdot (\mathbf{E}_i(0,t) + \mathbf{E}_\mathbf{r}(0,t)) = E_o \cos(\omega t - 0) - E_o \cos(\omega t + 0) = 0.$$

Now, what would $\mathbf{E}_r(x,t)$ be if the conducting reflector were not stationary on the $x = 0$ plane, but rather moving with a steady velocity v to the right, having a trajectory $x = vt$ as depicted in the margin?

- The answer of the question raised above is quite simple: We would have

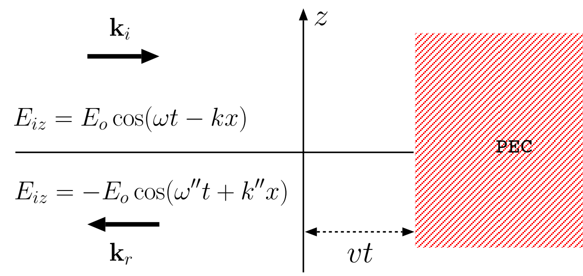
$$\mathbf{E}_r(x,t) = \hat{z} f(t + \frac{x}{c}),$$

where $f(t)$ is to be determined, so that

$$\hat{z} \cdot (\mathbf{E}_i(vt,t) + \mathbf{E_r}(vt,t)) = E_o \cos(\omega t - kvt) + f(t + \frac{vt}{c}) = 0,$$

because

1. $\mathbf{E}_r(x,t) = \hat{z} f(t + \frac{x}{c})$ is a viable (and the *only* viable) $\hat{z}$-polarized wave solution of Maxwell's equations propagating in the $-x$ direction in free space, and
2. the second equation above is the relevant boundary condition to be fulfilled on the surface of the moving reflector at every instant in time.



The boundary condition equation above implies that

$$f(t(1+\frac{v}{c})) = -E_o \cos(\omega t - kvt) = -E_o \cos(\omega t(1-\frac{v}{c})) = -E_o \cos(\omega \frac{1-\frac{v}{c}}{1+\frac{v}{c}} t(1+\frac{v}{c})).$$

Thus,

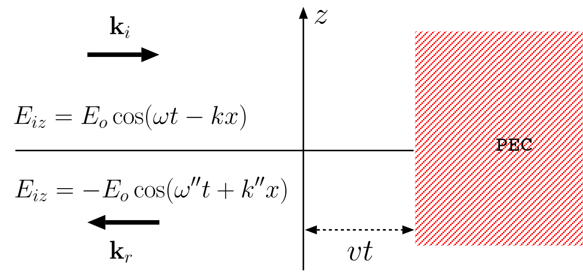
$$f(t) = -E_o \cos(\omega \frac{1-\frac{v}{c}}{1+\frac{v}{c}} t),$$

and

$$\mathbf{E}_r(x,t) = \hat{z} f(t+\frac{x}{c}) = -\hat{z} E_o \cos(\omega \frac{1-\frac{v}{c}}{1+\frac{v}{c}} (t+\frac{x}{c})) = -\hat{z} E_o \cos(\omega'' t + k'' x),$$

with

$$\omega'' = \omega\frac{1-\frac{v}{c}}{1+\frac{v}{c}} \quad \text{and} \quad k'' = \frac{\omega''}{c} = \frac{\omega}{c}\frac{1-\frac{v}{c}}{1+\frac{v}{c}} = k\frac{1-\frac{v}{c}}{1+\frac{v}{c}}.$$



With

$$\omega'' = \omega\frac{1-\frac{v}{c}}{1+\frac{v}{c}} \quad \text{and} \quad k'' = k\frac{1-\frac{v}{c}}{1+\frac{v}{c}}$$

the reflected wave

$$\mathbf{E}_r(x,t) = \hat{z} f(t + \frac{x}{c}) = -\hat{z} E_o \cos(\omega'' t + k'' x)$$

wave is clearly a co-sinusoid — just like the incident wave — but with *Doppler shifted* **frequency and wavenumbers** ω'' **and** k''**, respectively, caused by the motion of the reflector surface (as discussed below).** The result can also be used with negative v corresponding to a reflector moving to the left.

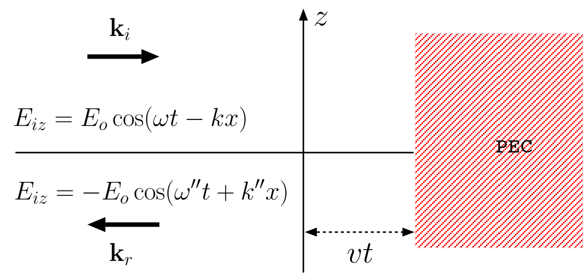
- The Doppler shift formulae given above are relativistically correct — that is, they are valid for all possible values of $\frac{v}{c}$ — even though we did not invoke any "relativistic argument" above.

This is true because *relativity* derives from the Maxwell's equations and the accompanying boundary conditions, and so *any* rigorous deduction derived from Maxwell's equations will be *by default* relativistically valid.

- Focusing next on the Doppler shifted frequency formula

$$\omega'' = \omega \frac{1 - \frac{v}{c}}{1 + \frac{v}{c}} = \omega \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}},$$

we can re-express ω'' as



$$\omega'' = \omega' \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \quad \text{with} \quad \omega' \equiv \omega \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}.$$

  – We now recognize the Doppler shifted frequency

$$\omega'' = \omega' \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}$$

of the reflected wave as a Doppler shifted version of the *wave frequency*

$$\omega' = \omega \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}$$

seen in the reflector frame, which is in turn a Doppler shifted version of the frequency ω of the in the incident wave field $\mathbf{E}_i(x, t)$ defined in the so-called[2] "lab frame".

This concludes our derivation of the relativistic Doppler shift formulae stated earlier on.

[2]By definition the frame where the "unprimed" frequency ω is observed is the *lab frame*; it can also be called the *unprimed frame*.

One-way Doppler shift:

When a TEM wave is observed to have a frequency ω in the lab frame (and wavenumber $k = \omega/c$ since we are concerned with free-space propagation at this point), the same TEM wave will appear to have a frequency ω' in a second reference frame which is in motion within the lab frame.

- The *one-way* Doppler shifted frequency ω' will be related to the lab-frame frequency ω as

$$\omega' = \omega\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

  if the moving observer has a velocity v in the lab frame defined to be positive in the direction of wave propagation (away from the wave source).

- For non-relativistic speeds such that $\frac{|v|}{c} \ll 1$ we have

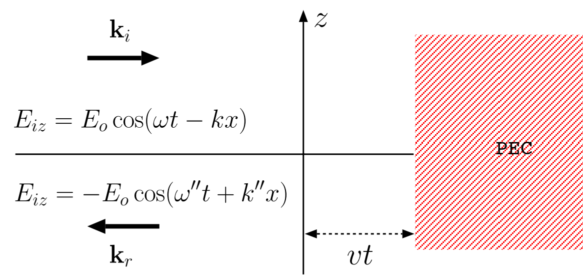
$$\omega' \approx \omega(1-\frac{v}{c}) = \omega - kv$$

  as already seen. This simplified Doppler formula is easy to understand since

$$\mathbf{E}_i(x,t) = \hat{z}E_o\cos(\omega t - kx)$$

  (see margin) implies that the incident field at the location $x = vt$ of the reflector must vary with time t as

$$\mathbf{E}_i(vt,t) = \hat{z}E_o\cos(\omega t - kvt) = \hat{z}E_o\cos((\omega - kv)t) = \hat{z}E_o\cos(\omega' t)$$



where

$$\omega' = \omega - kv$$

as obtained above[3].

- Relativistic Doppler shift equations given above are applicable for free-space propagation only — the reason is, a frequency independent propagation velocity was assumed in the derivation of ω''. The equations take modified forms[4] for propagation in material media. However, non-relativistic Doppler equations — as the time-rate-of-change of wave phase — are found to be valid in material media where ω/k is generally ω dependent.

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[3]An astute student may ask at this point: "how come $\mathbf{E}_i(vt, t) \propto \cos((\omega - kv)t)$ and not $\cos(\omega\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}t)$ if a rigorous application of electromagnetic solutions should produce relativistically accurate results (as claimed earlier on)?" This is the sort of question Albert Einstein asked to himself in his free time at work in a Swiss patent office and figured out that the rigorous conclusion ought to be

$$(\omega - kv)t = \omega\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}t' \;\Rightarrow\; t' = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}(1-\frac{v}{c})t = \sqrt{1-\frac{v^2}{c^2}}t,$$

where t' is the time kept by a clock attached to the reflecting surface. The fact that clocks in relative motion keep time at different rates — see the relativistic transformation formula between t' (measured on the reflector) and t (measured in the lab where the reflector is moving with velocity v) given in a footnote of Lecture 12 in ECE 329 notes — was one of the surprising results of the work Einstein published in 1905 under the title "On the Electrodynamics of Moving Bodies", popularly known as the *relativity* paper.

[4]The modified form

$$\omega' = \omega\frac{1-\frac{v}{c}n}{\sqrt{1-\frac{v^2}{c^2}}},$$

where $n \equiv \frac{c}{v_p}$ is the refractive index of the medium in terms of propagation speed $v_p = \frac{\omega}{k}$, hardly comes up in practice because relativistic velocities are rarely encountered within material media.