# 22 Dispersion and propagation in collisionless plasmas

- TEM plane wave propagation in homogeneous conducting media can be described in terms of wavenumbers and intrinsic impedances

$$k = \omega\sqrt{\mu(\epsilon + \frac{\sigma}{j\omega})} \equiv k' - jk'' \quad \text{and} \quad \eta = \sqrt{\frac{\mu}{\epsilon + \frac{\sigma}{j\omega}}} \equiv |\eta| e^{j\tau}$$

  as we have seen in Lecture 18.

  For **real valued** $\sigma$ these relations imply complex valued $k$ and $\eta$ as well as an $\omega$ *dependent* propagation velocity

$$v_p = \frac{\omega}{k'} = \frac{1}{\text{Re}\{\sqrt{\mu(\epsilon + \frac{\sigma}{j\omega})}\}}.$$

  **Having an $\omega$ dependent $v_p$ is a telltale sign that propagation of TEM waves in the medium will be *dispersive*, meaning that the shapes of TEM signals waveforms other than co-sinusoids will be distorted as a consequence of propagation — the distortion happens because different co-sinusoid components of the signal having different frequencies $\omega$ travel with different velocities $v_p$ and thus fall out of synchronism!**

- Dispersion in wave motions can be caused by a variety of reasons including the frequency dependence of the medium parameters as well as geometrical effects related to the dimensions of the propagation region in relation to a wavelength.

  - For an *ohmic* medium where $\sigma$ is real — such as seawater or copper — wave propagation is both *lossy* and dispersive.

- An important propagation medium known to be dispersive but *lossless* is the "**collisionless plasma**", an ionized gas in which collisions of the charge carriers (with one another) are negligibly small — a collisionless plasma provides an ideal setting to explore and understand the wave dispersion effects without having to deal with complications arising from losses and dissipation.

- A collisionless plasma is essentially a conducting medium with a purely imaginary conductivity $\sigma$ (or, equivalently, a dielectric with a relative permittivity less than one, as we will see).

  - To develop the conductivity model for a collisionless plasma we envision a region of volume in free-space containing $N$ free electrons per unit volume along with $N$ positive ions (e.g., $O^+$ in the ionized portions of the upper atmosphere) which are also free. Each of these free charge carriers with charge $q$ and mass $m$ respond to an alternating electric field with a phasor $\tilde{\mathbf{E}}$ as dictated by Newton's first law:

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} \quad \Rightarrow \quad mj\omega\tilde{\mathbf{v}} = q\tilde{\mathbf{E}}$$

where $\tilde{\mathbf{v}}$ denotes the phasor of particle velocity in sinusoidal steady-state. With $N$ electrons per unit volume, each carrying a charge $q = -e$ with a phasor velocity $\tilde{\mathbf{v}}$, we then have a phasor current density

$$\tilde{\mathbf{J}} = Nq\tilde{\mathbf{v}} = Nq(\frac{q\tilde{\mathbf{E}}}{mj\omega}) = -j\frac{Nq^2}{m\omega}\tilde{\mathbf{E}}$$

carried by free electrons only. Positive ions with much larger mass than the electrons will also carry a similar current density, but with a much smaller magnitude given the inverse mass dependence of the expression for $\tilde{\mathbf{J}}$. Hence, a reasonable model for a current density in a collisionless plasma is

$$\tilde{\mathbf{J}} = \sigma\tilde{\mathbf{E}} \quad \text{with a plasma conductivity} \quad \sigma = -j\frac{Ne^2}{m\omega}$$

where the contribution of ions is neglected.

The crucial result above is that conductivity is purely imaginary — the collisionless plasma is not a resistive but a reactive propagation medium!

**In the absence of collisions, kinetic energy of the charge carriers acquired from the wave field is not dissipated (lost) into heat, but instead returned to the wave field much like energy exchange in a circuit consisting of a source and an inductor.**

- Since TEM wave propagation in conducting media is the same as propagation in a dielectric with an effective permittivity

$$\epsilon + \frac{\sigma}{j\omega},$$

a collisionless *plasma* with $\epsilon = \epsilon_o$ and conductivity

$$\sigma = -j\frac{Ne^2}{m\omega}$$

can be treated like a dielectric with a permittivity

$$\epsilon_o + \frac{-j\frac{Ne^2}{m\omega}}{j\omega} = \epsilon_o(1 - \frac{\frac{Ne^2}{m\epsilon_o}}{\omega^2}) = \epsilon_o(1 - \frac{\omega_p^2}{\omega^2}),$$

where

$$\omega_p \equiv \sqrt{\frac{Ne^2}{m\epsilon_o}}$$

is known as the **plasma cutoff frequency** (or "plasma frequency" for short).

With the above definitions,

$$\epsilon_r = 1 - \frac{\omega_p^2}{\omega^2} \ \text{ and } \ n = \sqrt{\epsilon_r} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}},$$

are, respectively, the **relative permittivity** and **refractive index** in a plasma treated as a dielectric, where we also have $\mu = \mu_o$ (true because except for its free carriers, a plasma is essentially a vacuum).

Hence, in a plasma, TEM waves are described by a **wavenumber**

$$k = \omega\sqrt{\mu\epsilon} = \omega\sqrt{\mu_o\epsilon_o\epsilon_r} \equiv \omega\sqrt{\mu_o\epsilon_o}\sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}},$$

and an **intrinsic impedance**

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_o}{\epsilon_o\epsilon_r}} = \frac{\sqrt{\mu_o/\epsilon_o}}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} = \frac{\eta_o}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}},$$

respectively.

- The collisionless **plasma dispersion relation** for TEM waves, namely

  $$k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

  derived above governs the properties of TEM waves to be encountered in a plasma medium.

  It most significantly differs from the free-space dispersion relation

  $$k = \frac{\omega}{c}$$

  by exhibiting a non-linear relationship between $\omega$ and $k$.

  As a consequence,

  1. The **propagation velocity**

     $$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} = \frac{c}{n}$$

in a plasma is frequency dependent and a meaningful concept only for $\omega > \omega_p$ when $k$ is real valued (see next).

2. For $\omega < \omega_p$, we find a purely imaginary $k$ in which case $e^{-jkz}$ describes not a propagating wave but an **evanescent** one (for which $v_p$ is not a relevant concept).

3. The plasma **refractive index**

$$n = \frac{c}{v_p} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

is real valued but less than unity in the propagation regime $\omega > \omega_p$ and it is imaginary in the evanescence region $\omega < \omega_p$.

- Topics that remain to be examined over the next two lectures:

1. The distinction between **phase** and **group velocity** concepts in the regime $\omega > \omega_p$

2. Evanescent plasma waves in the $\omega < \omega_p$ regime and related tunneling phenomena.

- We close this lecture with a brief discussion of plasma frequency $\omega_p$.

- The parameter

$$\omega_p \equiv \sqrt{\frac{Ne^2}{m\epsilon_o}}$$

in the plasma dispersion relation has the dimension of frequency and grows with the square root of the electron density of the plasma. A useful formula for the plasma frequency is

$$\omega_p = 2\pi f_p \quad \text{with} \quad f_p \approx \sqrt{80.6 N}$$

where $f_p$ is quoted in Hz units when $N$ is entered in $\mathrm{m}^{-3}$ units.

- For example, for $N = 10^{12}\,\mathrm{m}^{-3}$ as in the Earth's ionosphere,

  $$f_p \approx \sqrt{80.6 \times 10^{12}} \approx 9 \times 10^6\,\mathrm{Hz} = 9\,\mathrm{MHz}$$

  and

  $$\omega_p \approx 18\pi\,\mathrm{Mrad/s}.$$

- A plasma frequency of $f_p \approx 9$ MHz will have a severe impact on EM waves in the ionosphere when the wave frequency $f = \frac{\omega}{2\pi}$ is close to 9 MHz.
- The effect of the plasma on the EM wave will be negligible in the ionosphere only when $f$ is many orders of magnitude larger than 9 MHz.

- The plasma frequency $\omega_p$ also has a direct interpretation in terms of plasma dynamics:

  - If all the electrons in a volume of plasma were pulled to one side of the volume, away from the positive ions within the same volume

— in analogy to a stretched spring — and then let go, the electron and ion populations would rush toward one another (because of electrostatic attraction) and then overshoot (because of inertia) and reverse their motions to establish a perpetual oscillation at the frequency $f_p$!

– The plasma frequency $f_p$ is a resonance frequency of the plasma seen as an elastic body.

**Example 1:** Consider an infinite homogeneous plasma with a plasma frequency of $f_p = 10$ MHz. Determine the wavelength or the penetration depth — whichever is relevant — of a TEM wave in the plasma produced by an infinite current sheet located at $x = 0$ plane, if the oscillation frequency of the current density is (a) $f = 20$ MHz, and (b) $f = 5$ MHz. Also comment whether the TEM wave is propagating or evanescent.

**Solution:** (a) We have, for $f = 20$ MHz

$$\begin{aligned} k &= \frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}} = \frac{2\pi\times 20\times 10^6}{3\times 10^8}\sqrt{1-(\frac{10}{20})^2} \\ &= \frac{40\pi}{300}\sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}40\pi}{600} = \frac{\sqrt{3}\pi}{15}\,\frac{\text{rad}}{\text{m}}. \end{aligned}$$

Hence, the TEM wave produced by the current sheet is "propagating" in that case (away from $x = 0$ surface on both sides) and its wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\frac{\sqrt{3}\pi}{15}} = \frac{30}{\sqrt{3}} = 10\sqrt{3}\,\text{m}.$$

(b) For $f = 5$ MHz

$$
\begin{aligned}
k &= \frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}} = \frac{2\pi \times 5 \times 10^6}{3 \times 10^8}\sqrt{1-(\frac{10}{5})^2} \\
&= \frac{10\pi}{300}\sqrt{1-4} = \pm j\frac{\sqrt{3}10\pi}{600} = \pm j\frac{\sqrt{3}\pi}{60}\frac{\text{rad}}{\text{m}}
\end{aligned}
$$

where the sign giving rise to the decaying wave away from its source should be employed. In this case the TEM wave is "evanescent" and its attenuation constant is

$$|k| = \frac{\sqrt{3}\pi}{60}\frac{\text{Np}}{\text{m}}.$$

The corresponding penetration distance (distance over which the wave amplitude is reduced by an exponential factor of $e^{-1}$) is

$$\delta = \frac{1}{|k|} = \frac{60}{\sqrt{3}\pi} = \frac{\sqrt{3}}{\pi}20\,\text{m}.$$

**Example 2:** In Example 1, part (b) what is the attenuation rate of the evanescent wave in units of dB/m?

**Solution:** The evanescent field in Example 1 part (b) will vary as

$$\tilde{\mathbf{E}}(x) = \hat{p}E_o e^{-|k|x}$$

in $x > 0$ region, where $\hat{p}$ is a unit vector perpendicular to $\hat{x}$, $E_o$ is the field strength at $x = 0$. Also,

$$|k| = \frac{\sqrt{3}\pi}{60}\frac{\text{Np}}{\text{m}}.$$

from the solution of part (b). Therefore, we have

$$\frac{|\tilde{\mathbf{E}}(x=0)|}{|\tilde{\mathbf{E}}(x=1)\,\mathrm{m}|} = \frac{1}{e^{-|k|}} = e^{|k|}$$

and

$$\begin{aligned} 20\log_{10}\frac{|\tilde{\mathbf{E}}(x=0)|}{|\tilde{\mathbf{E}}(x=1)\,\mathrm{m}|} &= 20\log_{10} e^{|k|} = |k| 20\log_{10} e \\ &= \frac{\sqrt{3}\pi}{60}\, 20\log_{10} e = \frac{\pi}{\sqrt{3}}\log_{10} e \approx \frac{\pi}{\sqrt{3}} 0.434 \\ &\approx 0.788\,\frac{\mathrm{dB}}{\mathrm{m}} \end{aligned}$$

which is the attenuation rate of the evanescent field in dB/m units.

Attenuation in dB/m is expressed as the 20 log of the amplitude ratio across a one meter distance (as we have done above), or, equivalently, as 10 log of the "power" ratio across a one meter distance.