23 Phase and group velocities and delays

- Propagation velocity

$$v_p = \frac{\omega}{k}$$

  of a co-sinusoid field component

$$\cos(\omega t - kz) \quad \Leftrightarrow \quad e^{-jkz}$$

  is also known as **phase velocity**, because v_p as defined above, corresponds to the speed with which constant phase points (e.g., zero-crossings of the field) move.

- If the phase velocity is ω dependent — as in dispersive media — then field components (e.g., E_x, H_y, etc.), which are the superpositions of co-sinusoids with different frequencies (two, three, several, countless), can in general be described in terms of an **envelope** function and a **carrier** function (recall AM modulation from ECE 210), each having its own and distinct velocity.

  – The propagation velocity of the envelope is known as **group velocity** and it can be calculated as

$$v_g = \frac{\partial \omega}{\partial k}$$

    once the dispersion relation relating k to ω is available.

- The propagation velocity of the carrier is simply a phase velocity

$$v_p = \frac{\omega}{k},$$

where we use the carrier frequency ω_o for frequency ω, and the carrier wavenumber k_o for wavenumber k as illustrated below.

A simple example: Consider the superposition

$$f(z,t) = \cos(\omega_1 t - k_1 z) + \cos(\omega_2 t - k_2 z)$$

where wavenumbers k_1 and k_2 depend on frequencies $\omega_1$and ω_2 as described by some dispersion relation (e.g., the plasma dispersion relation). Using some trig identities we can re-write $f(z,t)$ as

$$f(z,t) = 2\cos(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}z)\cos(\omega_o t - k_o z)$$

where

$$\omega_o \equiv \frac{\omega_1 + \omega_2}{2}, \quad k_o \equiv \frac{k_1 + k_2}{2}, \quad \Delta\omega = \omega_2 - \omega_1, \quad \Delta k = k_2 - k_1.$$

- **Verification:** Given the above definitions,

$$\omega_{1,2} = \omega_o \mp \frac{\Delta\omega}{2} \text{ and } k_{1,2} = k_o \mp \frac{\Delta k}{2},$$

and so

$$f(z,t) = \cos(\omega_1 t - k_1 z) + \cos(\omega_2 t - k_2 z)$$

$$
\begin{aligned}
&= \cos(\omega_o t - k_o z + \frac{\Delta\omega t - \Delta k z}{2}) + \cos(\omega_o t - k_o z - \frac{\Delta\omega t - \Delta k z}{2}) \\
&= \cos(\omega_o t - k_o z)\cos(\frac{\Delta\omega t - \Delta k z}{2}) - \sin(\omega_o t - k_o z)\sin(\frac{\Delta\omega t - \Delta k z}{2}) \\
&+ \cos(\omega_o t - k_o z)\cos(\frac{\Delta\omega t - \Delta k z}{2}) + \sin(\omega_o t - k_o z)\sin(\frac{\Delta\omega t - \Delta k z}{2}) \\
&= \underbrace{2\cos(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}z)}_{\text{envelope}}\ \underbrace{\cos(\omega_o t - k_o z)}_{\text{carrier}}.
\end{aligned}
$$

In this simplest possible example of superpositioned co-sinusoids (simplest because we only used two components instead of many), both the envelope function and the carrier function are co-sinusoids.

Assuming that $\Delta\omega \ll \omega_1, \omega_2$, the carrier function $\cos(\omega_o t - k_o z)$ is a co-sinusoid within the same "frequency band" as the superposed co-sinusoids, while the envelope function $2\cos(\frac{\Delta\omega t - \Delta k z}{2})$ is a low-frequency co-sinusoid residing (in frequency space) outside the signal band.

With that distinction in mind, we identify the propagation velocities of the carrier and envelope functions as the phase and group velocities of composite waveform $f(z,t)$ — the phase velocity (describing the carrier motion) is

$$v_p = \frac{\omega_o}{k_o},$$

whereas the group velocity (describing the envelope motion) is

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{\omega_2 - \omega_1}{k_2 - k_1}.$$

Example 1: Consider the case

$$\Delta\omega = \frac{\omega_o}{10} \quad \text{and} \quad \Delta k = \frac{k_o}{5}.$$

In that case

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{\omega_o/10}{k_o/5} = \frac{1}{2}\frac{\omega_o}{k_o} = \frac{1}{2}v_p,$$

a waveform with half as large a group velocity as the phase velocity — in such a waveform, the zero-crossings of the carrier will march through the envelope as demonstrated by an animation on the web site.

Example 2: Determine the group velocity

$$v_g = \frac{\Delta\omega}{\Delta k}$$

of the sum of two co-sinusoidal waves propagating in z direction if $\omega_1 = 99$ rad/s, $\omega_2 = 101$ rad/s and the dispersion relation is

$$\omega = k^2.$$

Solution: We can solve this problem by first obtaining $k_{1,2} = \sqrt{\omega_{1,2}}$, and then dividing

$$\Delta\omega = \omega_2 - \omega_1$$

by

$$\Delta k = \sqrt{\omega_2} - \sqrt{\omega_1}.$$

Alternatively, we can approximate $\Delta\omega/\Delta k$ by the

$$\text{partial derivative } \partial\omega/\partial k \text{ evaluated at } \omega_o = 100\,\text{rad/s}$$

which is at the center of the frequency band flanked by ω_1 and ω_2.

Both approaches will give about the same result since $\Delta\omega \ll \omega_o$ and the slope $\partial\omega/\partial k$ of the ω versus k curve at $\omega = \omega_o$ is nearly the same as the ratio $\Delta\omega/\Delta k$.

Using the second method, we note

$$\omega = k^2 \;\; \Rightarrow \;\; \frac{\partial\omega}{\partial k} = 2k = 2\sqrt{\omega}.$$

Therefore, the group velocity of the sum is

$$v_g = \frac{\Delta\omega}{\Delta k} \approx \frac{\partial\omega}{\partial k} = 2\sqrt{\omega} = 20\,\frac{\text{m}}{\text{s}}$$

after evaluating at $\omega = \omega_o = 100$ rad/s. You should compare our result above with the exact value

$$\frac{\Delta\omega}{\Delta k} = \frac{\omega_2 - \omega_1}{\sqrt{\omega_2} - \sqrt{\omega_1}}$$

to convince yourself that both approaches give approximately the same result.

Example 3: What is the phase velocity of the sum signal in Example 2.

Solution: The phase velocity of the sum signal is

$$v_p = \frac{\omega_o}{k_o}$$

where

$$\omega_o \equiv \frac{\omega_1 + \omega_2}{2}, \quad k_o \equiv \frac{k_1 + k_2}{2}, \quad k = \sqrt{\omega}.$$

This is well approximated by

$$v_p = \frac{\omega_o}{\sqrt{\omega_o}} = \frac{100}{10} = 10\,\frac{\mathrm{m}}{\mathrm{s}}.$$

The exact value can be obtained as

$$v_p = \frac{\omega_o}{k_o} = \frac{\omega_1 + \omega_2}{k_1 + k_2} = \frac{\omega_1 + \omega_2}{\sqrt{\omega_1} + \sqrt{\omega_2}}.$$

- Practical signals used in communication applications are more complicated than just the sum of two-co-sinusoids. In general, we are concerned with the superposition of a continuum of co-sinusoids over finite frequency bands $\Delta\omega$. How do we then define the wave envelope and the carrier in such cases and extend the notion of phase and group velocities introduced above? This question is addressed next:

- Consider a sum of many monochromatic waves of frequencies ω_m in a band $\Delta\omega$ centered about a frequency ω_o — such a sum can be represented as

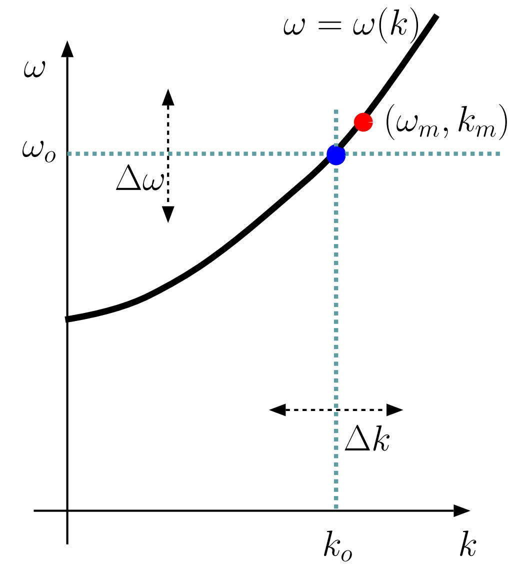
$$\sum_m \mathrm{Re}\{F_m e^{j(\omega_m t - k_m z)}\}$$

where F_m are the individual wave amplitudes. Introducing

$$\omega_m = \omega_o + \Delta\omega_m \text{ and } k_m = k_o + \Delta k_m,$$

we can re-write the same sum as

$$\mathrm{Re}\{e^{j(\omega_o t - k_o z)} \sum_m F_m e^{j\Delta\omega_m (t - \frac{z}{\Delta\omega_m/\Delta k_m})}\}.$$



Suppose that the band of frequencies $\Delta\omega$ containing all the components ω_m is sufficiently small so that the ratio

$$\frac{\Delta\omega_m}{\Delta k_m} \approx \frac{\partial\omega}{\partial k}_{|\omega=\omega_o}$$

is independent of index m.

In that case the sum above reduces to

$$\mathrm{Re}\{e^{j(\omega_o t - k_o z)} f(t - \frac{z}{v_g})\}$$

with

$$f(t) = \sum_m F_m e^{j\Delta\omega_m t} \quad \text{Envelope function}$$

and

$$v_g = \frac{\partial \omega}{\partial k}_{|\omega=\omega_o} \quad \text{Group velocity.}$$

- The above result indicates that a wave signal

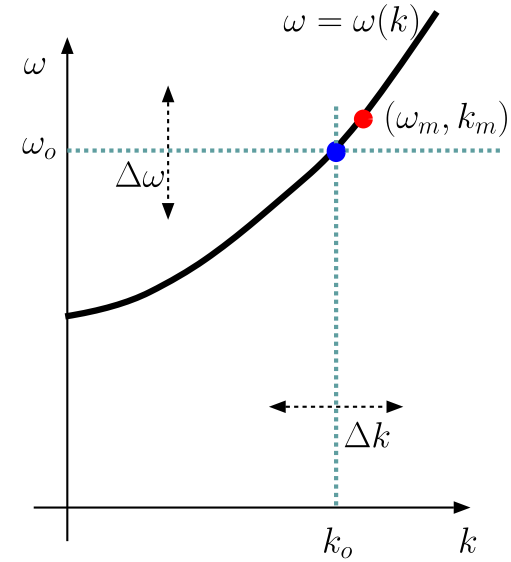
$$s(0,t) = \mathrm{Re}\{e^{j\omega_o t} f(t)\} = |f(t)| \cos(\omega_o t + \angle f(t))$$

observed at a location $z = 0$ will be observed at an arbitrary $z > 0$ as

$$s(z,t) = \mathrm{Re}\{e^{j(\omega_o t - k_o z)} f(t - \frac{z}{v_g})\} = |f(t - \frac{z}{v_g})| \cos(\omega_o t - k_o z + \angle f(t - \frac{z}{v_g})).$$

Such a signal[1] would be called an

1. **AM signal** for the case $\angle f(t) = 0$ — purely real $f(t)$, requiring $F_{-m} = F_m^*$ for $\Delta\omega_m = m\delta\omega$, with $m = 0, \pm 1, \pm 2, \cdots$,
2. **Phase modulated (PM) signal** for $|f(t)| = $ const.— $f(t) \propto e^{j\phi(t)}$, requiring $F_{-m} = -F_m^*$ and $|F_m| \ll 1$ for $m = \pm 1, \pm 2, \cdots$.



The important point is, the modulation $|f(t)|$ and/or $\angle f(t)$ of the AM and/or PM signal will travel with the group velocity

$$v_g = \frac{\partial \omega}{\partial k}_{|\omega=\omega_o}.$$

[1] Also, the same results apply in the continuum limit of $\delta\omega \to 0$ in which case the sum defining $f(t)$ in terms of Fourier coefficients F_m reduce to an integral defining $f(t)$ in terms its Fourier transform $F(\omega)$.

- The propagation of **narrowband signals** for which the above derivation of v_g is well justified — bandwidth $\Delta\omega \ll \omega_o$ — is therefore well described by the phase velocity (for the carrier) and the group velocity (for the modulation envelope and/or phase) concepts.

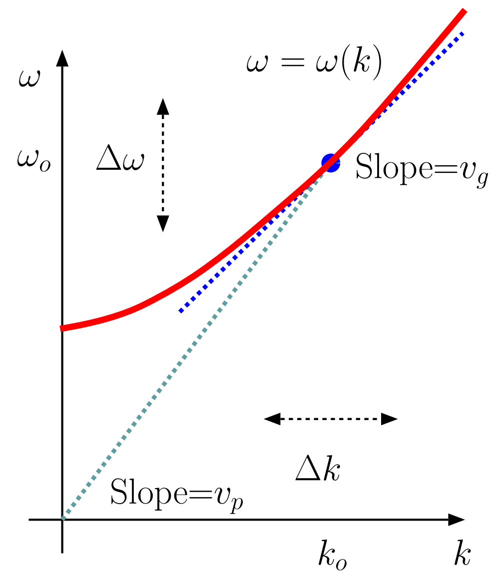
  – However, for **broadband signals** where $\Delta\omega \sim \omega_o$ the constancy of
  $$\frac{\Delta\omega_m}{\Delta k_m}$$
  over the entire frequency band $\Delta\omega$ will not hold, and it may be necessary to define a set of frequency dependent group velocities associated with sub-bands of $\Delta\omega$ (see HW).

- In general, the computation of the phase and group velocities of narrowband signals requires the knowledge of pertinent dispersion relation, the algebraic relationship between the wave frequency ω and wavenumber k.

- When the dispersion relation is known, it is useful to display it in the form of a ω versus k plot as shown in the margin.

  – If the plot is a straight line then the waves are dispersionless and $v_g = v_p$.

  – However, if the plot is curved (like shown in the margin), then the waves are dispersive and the phase and group velocities v_p and v_g need to be computed separately.



Note that:

Phase velocity **v_p is the slope of the line from the origin to the dispersion curve at the band center.**

Group velocity **v_g is the slope of the dispersion curve itself at the band center.**

- The dispersion relation

$$k = \frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}} \;\Rightarrow\; c^2k^2 = \omega^2 - \omega_p^2 \;\Rightarrow\; \omega = \sqrt{c^2k^2+\omega_p^2}$$

for the collisionless plasma has a dispersion curve resembling the one shown in the margin — waves in a plasma are clearly dispersive.

To obtain the plasma group velocity we take the partial derivative of the plasma dispersion formula

$$c^2k^2 = \omega^2 - \omega_p^2$$

on both sides with respect to variable k, which leads to

$$\frac{\partial}{\partial k}(c^2k^2 = \omega^2 - \omega_p^2) \;\Rightarrow\; c^2 2k = 2\omega\frac{\partial\omega}{\partial k} \;\Rightarrow\; \frac{\omega}{k}\frac{\partial\omega}{\partial k} = c^2.$$

Since

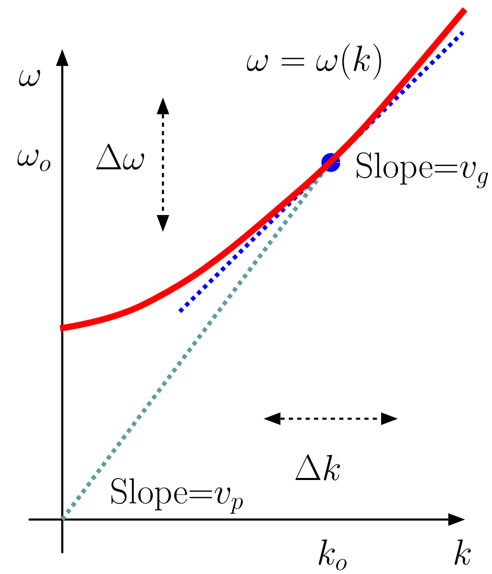
$$v_g = \frac{\partial\omega}{\partial k} \quad \text{and} \quad v_p \equiv \frac{\omega}{k},$$

this result indicates that in a plasma

$$v_g v_p = c^2$$

with the explicit formulas of

$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1-\frac{\omega_p^2}{\omega^2}}} \quad \text{and} \quad v_g = c\sqrt{1-\frac{\omega_p^2}{\omega^2}}.$$



Note that:

Phase velocity **v_p is the slope of the line from the origin to the dispersion curve at the band center.**

Group velocity **v_g is the slope of the dispersion curve itself at the band center.**

Note that the phase velocity in the plasma exceeds c at all $\omega > \omega_p$, whereas the group velocity is bounded by c.

Einstein's speed limit of c for motions in the universe is only meant to apply to energy, mass, and information transport — that list does not include the phase velocity, since the phase velocity of an unmodulated carrier is not pertinent for the transfer of energy or mass or information across space.

- A distant light bulb can be lit by sending an electric field pulse with an envelope which will travel the intervening distance at the group velocity of the propagation medium. The light bulb gets turned on only after the pulse envelope arrives at its location, independent of how fast (or slow) the pulse carrier moves. Energy moves with the group velocity[2].

- In general, depending on the dispersion relation, it is possible to have $v_g < v_p$ as well as $v_g > v_p$.

- Also, as just mentioned, it is possible to have $v_p < c$ as well as $v_p > c$ (as dictated by the relevant dispersion relation).

- However, $v_g > c$ is never possible for any wave motion — if a group velocity calculation indicates $v_g > c$ in some setting, you can be sure that the dispersion relation used for v_g calculation is invalid in that

[2]A rigorous proof that energy is transported with velocity v_g in linear and dispersive media can be found in Bers, *Am. J. Phys.*, **68**, 482 (2000).

setting and/or the dispersion curve has a shape that precludes the applicability of the narrowband signal model developed in this lecture.