25 Parallel-plate waveguides — TE$_m$ modes

With this lecture we start our study of guided waves and resonant cavities.
In ECE 329 you were already exposed to guided TEM wave propagation in two-wire transmission line systems.
Here we will study TE and TM mode propagation on parallel-plate transmission-lines and hollow waveguides.

While guided TEM modes are dispersion-free and propagate at all frequencies, TE and TM modes are dispersive and exhibit frequency-dependent cutoff.

- Consider a TE polarized incident field

$$\tilde{\mathbf{E}}_i = \hat{y} E_o e^{-j(-k_x x + k_z z)}$$

incident on a conducting plate on $x = 0$ plane as depicted in the margin so that a reflected wave

$$\tilde{\mathbf{E}}_r = -\hat{y} E_o e^{-j(k_x x + k_z z)}$$

is produced to ascertain $\hat{x} \times (\tilde{\mathbf{E}}_i + \tilde{\mathbf{E}}_r) = 0$ on $x = 0$ plane. In these expressions

$$k_x^2 + k_z^2 = k^2 = \omega^2 \mu_o \epsilon_o$$

assuming that the plate is embedded in free space (otherwise use μ and ϵ, instead).

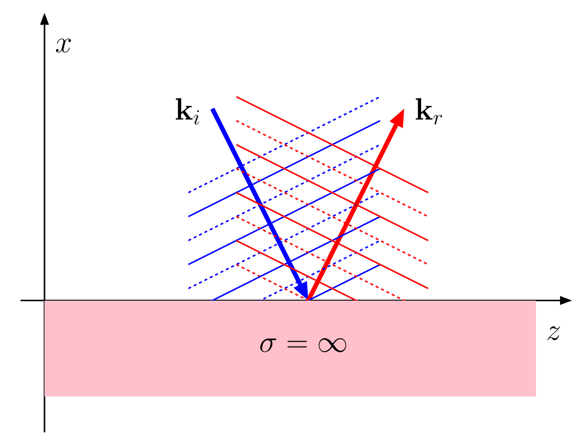
- The incident and reflected waves will then produce a **total field**

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_i + \tilde{\mathbf{E}}_r = \hat{y} E_o e^{-jk_z z}(e^{jk_x x} - e^{-jk_x x}) = 2j\hat{y} E_o e^{-jk_z z} \sin(k_x x)$$

in $z > 0$ region which propagates in z-direction with a phase velocity

$$v_{pz} = \frac{\omega}{k_z}$$

and "stands" in the x-direction.



Intersections of solid and dashed wavefronts of the incident and reflected waves demark the locations of the nulls of the total y-directed electric field.

- The standing wave in the x-direction is characterized by a magnitude

$$|\tilde{\mathbf{E}}| \propto |\sin(k_x x)|$$

which has nulls and maxima repeating along the x-direction at intervals

$$\frac{\lambda_x}{2} \quad \text{where} \quad \lambda_x \equiv \frac{2\pi}{k_x}.$$

  – Since there is an electric field null at $x = 0$, there additional nulls are located at

$$x = m\frac{\lambda_x}{2}, \quad \text{with} \quad m = 1, 2, \cdots$$

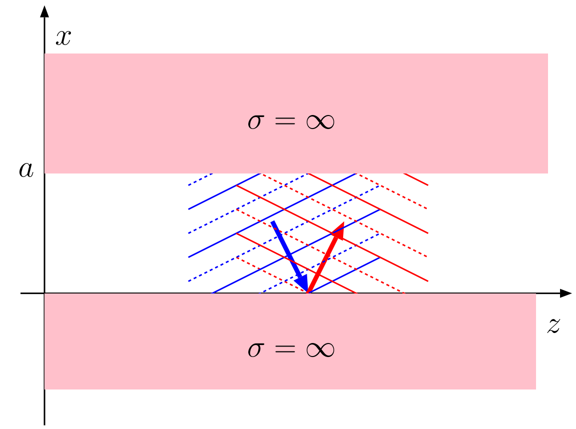
  Now, if a second plate were placed at $x = a$ (as shown in the margin), such that

$$a = m\frac{\lambda_x}{2} = \frac{m\pi}{k_x} \quad m = 1, 2, \cdots$$

  the standing wave pattern would not be perturbed by the new plate "fitting" the null-tangential field surface, providing us with the "guiding condition"

$$k_x = \frac{m\pi}{a}, \quad m = 1, 2, \cdots$$

  of TE-polarized wave fields by a pair of conducting plate separated by a distance a.



The y-directed total electric field is zero at x=0 and x=a surfaces of the guide formed by the conducting plates and exhibit maximum magnitude at the intersections of solid or dashed wavefront pairs.

Note the 180 degree reversals in the reflected phase fronts on both surfaces (top and bottom)as required by a reflection coefficient of -1.

– $m = 0$ is not allowed since it leads to $k_x = 0$ and consequently to $\tilde{\mathbf{E}} = 0$ for all x — a trivial case of having no field at all!

- For each allowed value of $m = 1, 2, \cdots$ we then have guided "TE$_m$ mode" waves having the electric field phasors

$$\tilde{\mathbf{E}} = 2j\hat{y}E_o e^{-jk_z z}\sin(k_x x)$$

in the region $0 < x < a$ with

$$k_x = \frac{m\pi}{a}$$

and

$$k_z = \sqrt{k^2 - k_x^2} = k\sqrt{1 - \frac{k_x^2}{k^2}} = \frac{\omega}{c}\sqrt{1 - \frac{k_x^2 c^2}{\omega^2}} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_c^2}{\omega^2}},$$

where

$$\omega_c \equiv k_x c = \frac{m\pi c}{a}$$

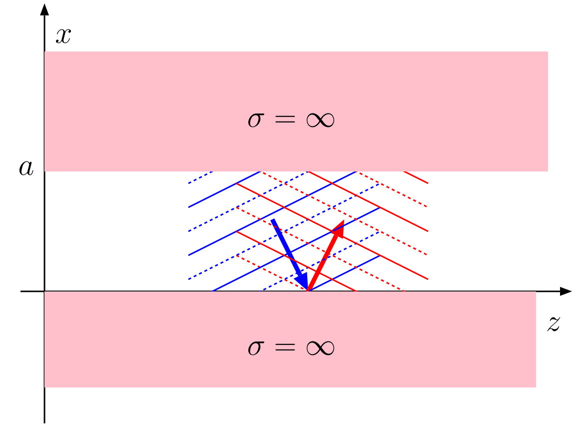
is known as cutoff frequency of TE$_m$ mode.

– The corresponding field in the time domain is obtained by multiplying the phasor with $e^{j\omega t}$ and taking the real part, yielding

$$\mathbf{E}(x,t) = -2\hat{y}E_o \sin(\omega t - k_z z)\sin(k_x x).$$

Since propagation of the TE$_m$ mode field is controlled by k_z, the relationship

$$k_z = \frac{\omega}{c}\sqrt{1 - \frac{\omega_c^2}{\omega^2}} = \frac{\omega}{c}\sqrt{1 - \frac{f_c^2}{f^2}},$$



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where

$$\omega_c = k_x c = \frac{m\pi c}{a} \quad \text{and} \quad f_c \equiv \frac{\omega_c}{2\pi} = \frac{mc}{2a},$$

is the **dispersion relation** of the TE_m mode, from which it follows that:

1. **Propagation** takes place if $f > f_c = \frac{mc}{2a}$, and **evanescence** otherwise, and
2. Phase and group velocities

$$v_{pz} \equiv \frac{\omega}{k_z} = \frac{c}{\sqrt{1-\frac{f_c^2}{f^2}}} \quad \text{and} \quad v_g \equiv \frac{\partial \omega}{\partial k_z} = c\sqrt{1-\frac{f_c^2}{f^2}}$$

in analogy with plasma dispersion (identical except for the interchange of ω_c with ω_p).

- The component TEM waves that satisfy the condition

$$\lambda f = c \quad \Leftrightarrow \quad \frac{\omega}{k} = c$$

and superpose to form the TE_m mode have a wavelength, for

$$f = f_c = \frac{mc}{2a},$$

given by

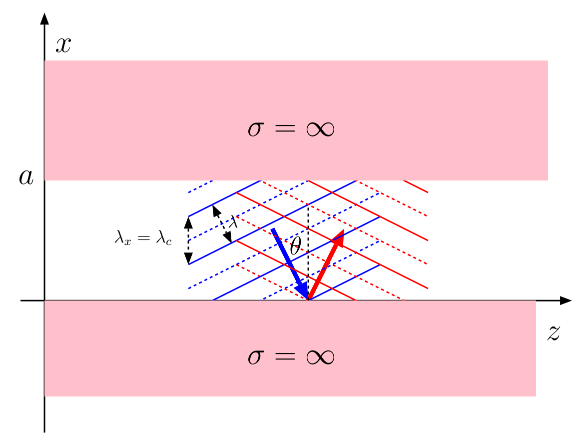
$$\lambda = \frac{c}{f} = \frac{c}{f_c} = \frac{c}{\frac{mc}{2a}} = \frac{2a}{m} \equiv \lambda_c.$$

Example:
for $a = 3$ cm, $m = 1$ implies

$$\begin{aligned} f_c &= \frac{mc}{2a} = \frac{3\times 10^8}{2\times 0.03} \\ &= 5\times 10^9\,\text{Hz} \\ &= 5\,\text{GHz} \end{aligned}$$

for TE_1mode.
The cutoff frequency for TE_2mode is 10 GHz, for TE_3mode is 15 GHz, and so on.

A signal with 11 GHz frequency will propagate in TE_1 and TE_2mode, but will be evanescent in $\text{TE}_3$and higher order modes.

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Note that cutoff-wavelength=2a/m is also
the "trace wavelength" in x-direction.
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Consequently, the TE_m mode dispersion relation can also be cast as

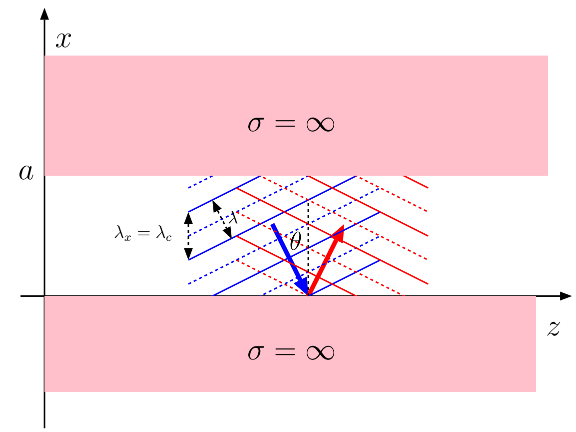
$$k_z = \frac{\omega}{c}\sqrt{1 - \frac{f_c^2}{f^2}} = \frac{\omega}{c}\sqrt{1 - \frac{c^2}{\lambda_c^2 f^2}} = \frac{\omega}{c}\sqrt{1 - \frac{\lambda^2}{\lambda_c^2}};$$

note that:

1. The "cutoff wavelength"
$$\lambda_c = \frac{2a}{m}$$
can be remembered to be "twice the guide width a divided by the mode number m",

2. The factor $\frac{\lambda}{\lambda_c}$ can be used to replace the factor $\frac{f_c}{f}$ that appears in all of the expressions given above (and below).

3. Finally, since $k_x = \frac{m\pi}{a}$, we have
$$\lambda_c = \frac{2a}{m} = \frac{2\pi}{k_x} = \lambda_x$$
at any frequency f.



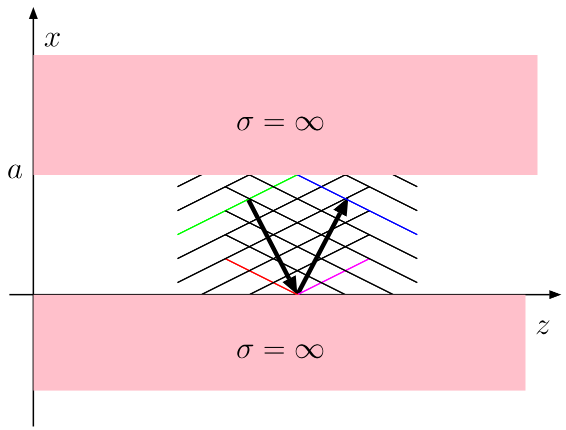
Note that cutoff-wavelength=2a/m is also the "trace wavelength" in x-direction.

- Each permissible k_x (or mode) at a given operation frequency ω is associated with an incidence *and* reflection angle θ (see margin) of the component TEM waves superposing, which is given by

$$\cos\theta = \frac{k_x}{k} = \frac{\lambda}{\lambda_x} = \frac{\lambda}{\lambda_c} = \frac{f_c}{f} \;\Rightarrow \theta = \cos^{-1}\frac{\lambda}{\lambda_c} = \cos^{-1}\frac{f_c}{f},$$

indicating that as $f \to f_c$, $\theta \to 0$, that is at $f = f_c$ (at cutoff), the field consists of plane waves bouncing back and forth between the plates at $x = 0$ and $x = a$ at normal incidence (see margin).

- Component waves superposing to constitute the guided field can be viewed as reflections of one another from the guide plates, produced (self-consistently) by surface currents induced on the conducting walls of the guide at $x = 0$ and $x = a$.

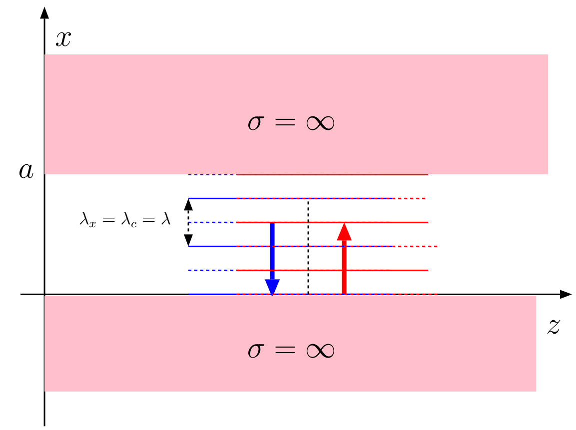


Below, we re-derive the "guidance condition"

$$k_x = \frac{m\pi}{a}$$

for TE$_m$ modes — with reference to the diagram shown above — by requiring "a phase consistency" between the reflected wave-pairs "responsible for one another":

- Note the four of the phase fronts in the diagram above (repeated in the margin) belonging, in pairs, to the "upgoing" and "downgoing" component TEM waves of y-polarized electric field have been highlighted by

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At cutoff (f=f_c) we have k_z=0 and thus
TEM waves bouncing between the plates in
exclusively x direction, carrying no energy
in z-direction.
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Guide wavelength: The component TEM waves that superpose to form the TE$_m$ mode solutions have a wavelength

$$\lambda = \frac{2\pi}{k}$$

as usual. We define

$$\lambda_z \equiv \frac{2\pi}{k_z} = \frac{2\pi}{k\sqrt{1 - \frac{f_c^2}{f^2}}} = \frac{\lambda}{\sqrt{1 - \frac{f_c^2}{f^2}}}$$

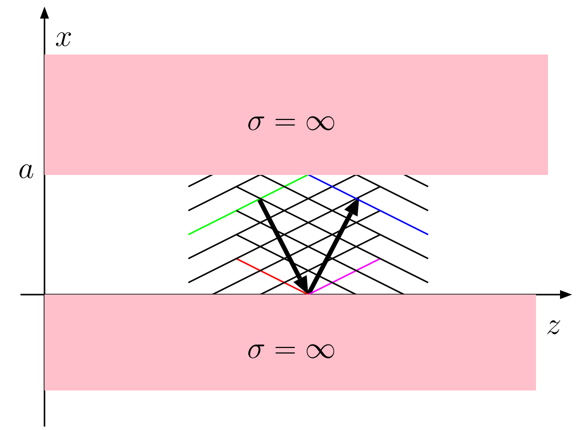
to be the *guide wavelength* λ_g. Note the distinction between

$$\lambda_g = \frac{2\pi}{k_z} = \lambda_z$$

and

$$\lambda_c = \frac{2\pi}{k_x} = \lambda_x.$$

the colors red and blue, and green and magenta, respectively. Indicating the phases on these singled out phase fronts as ϕ_r, ϕ_b, ϕ_g, and ϕ_m, we note in succession that



–
$$\phi_b = \phi_r - k_x a$$

because of phase delay $k_x a$ in the upgoing wave over an excursion by a along the x-axis,

–
$$\phi_g = \phi_b + \angle\Gamma,$$

where Γ=-1 is the reflection coefficient on the upper plate converting the upgoing electric field wave into a downgoing wave (to which ϕ_g belongs),

–
$$\phi_m = \phi_g - k_x a$$

because of phase delay $k_x a$ in the downgoing wave over an excursion by a along the x-axis,

–
$$\phi_r = \phi_m + \angle\Gamma,$$

where Γ=-1 is the reflection coefficient on the lower plate converting the downgoing wave into an upgoing wave (to which ϕ_r belongs).

– Now back-substituting in the expression for ϕ_r the expressions for ϕ_m, ϕ_g, ϕ_b in succession, we find that

$$\phi_r = \phi_r - 2(k_x a - \angle\Gamma)$$

Requiring $\angle\Gamma$ to lie in the range $-\pi < \angle\Gamma \leq \pi$, and given the inherent 2π ambiguities permitted for a wave phase (because of the non-discriminatory response of co-sinusoids to phases differing by integer multiples of 2π), this condition implies that we can take

$$k_x a - \angle\Gamma = n\pi, \quad n = 0, 1, 2, \cdots$$

since $k_x a$ is *by definition* non-negative. With

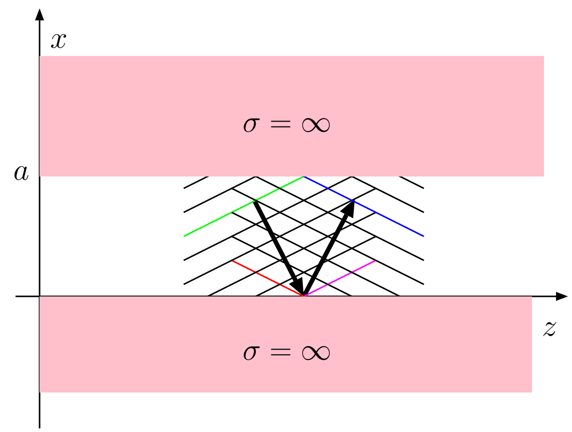
$$\Gamma = -1 \quad \Rightarrow \quad \angle\Gamma = \pi,$$

the condition reduces to

$$k_x a = (n+1)\pi = m\pi, \quad m = 1, 2, 3, \cdots$$

for TE_m modes, which is the same condition we obtained by using uniqueness arguments in connection with the standing-wave field solution earlier on.

$$\begin{aligned} \phi_b &= \phi_r - k_x a, \\ \phi_g &= \phi_b + \angle\Gamma, \\ \phi_m &= \phi_g - k_x a, \\ \phi_r &= \phi_m + \angle\Gamma. \end{aligned}$$



We will utilize the relation

$$k_x a - \angle\Gamma = n\pi, \quad n = 0, 1, 2, \cdots,$$

with the constraint that $-\pi < \angle\Gamma \leq \pi$ later on when we will study dielectric-slab waveguides — in that case we will have a TIR ($\theta_1 > \theta_c$) related reflection coefficient Γ with a θ_1 dependent phase angle $\angle\Gamma$.

- There is an even easier way of obtaining the same guidance condition using the following steps: The guided TE_m modes propagating in z direction are superpositions of incident and reflected TEM plane waves

$$\tilde{\mathbf{E}}_i = \hat{y}E_o e^{-j(-k_x x + k_z z)}$$

and

$$\tilde{\mathbf{E}}_r = \hat{y}E_o \Gamma e^{-j(k_x x + k_z z)}$$

where $\Gamma = \Gamma_\perp = -1$ is the TE-mode reflection coefficient appropriate for an air-PEC interface. Since for the permissible guided modes, TEM wave $\tilde{\mathbf{E}}_r$ gets reflected (once more) *at* $x = a$ to become $\tilde{\mathbf{E}}_i$ at the same location, it is necessary that

$$(\hat{y}E_o \Gamma e^{-j(k_x a + k_z z)})\Gamma = \hat{y}E_o e^{-j(-k_x a + k_z z)} e^{-j2\pi n}$$

for any integer n, i.e.,

$$|\Gamma|^2 e^{j2\angle\Gamma} e^{-jk_x a} = e^{jk_x a} e^{-j2\pi n}.$$

This is possible if and only if $|\Gamma| = 1$ and

$$2\angle\Gamma - k_x a = k_x a - 2\pi n \quad \Rightarrow \quad k_x a = \angle\Gamma + n\pi = m\pi,$$

the same guidance condition that we obtained earlier on.

⇐ Pay close attention to this method.

Next lecture we will use it to derive TM modes.

Example 1: TE$_m$ mode fields have transverse electric field phasors

$$\tilde{\mathbf{E}} = 2j\hat{y}E_o e^{-jk_z z}\sin(k_x x)$$

satisfying the zero-tangential field conditions at $x = 0$ and $x = a$ planes with

$$k_x = \frac{m\pi}{a} \text{ and } k_z = \frac{\omega}{c}\sqrt{1 - \frac{f_c^2}{f^2}}$$

where

$$f_c = \frac{mc}{2a}.$$

(a) Determine the magnetic field intensity phasor $\tilde{\mathbf{H}}$ for TE$_m$ mode waves. (b) Also determine $\eta_{TE} \equiv \frac{E_y}{-H_x}$, which is the effective guide impedance for TE$_m$ mode.

Solution: (a) Using Faraday's law, we have

$$\begin{aligned}
\tilde{\mathbf{H}} &= \frac{\nabla \times \tilde{\mathbf{E}}}{-j\omega\mu_o} = \frac{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}}{-j\omega\mu_o} = \frac{-\hat{x}\frac{\partial E_y}{\partial z} + \hat{z}\frac{\partial E_y}{\partial x}}{-j\omega\mu_o} \\
&= -\frac{2E_o}{\omega\mu_o}(\hat{x}(jk_z\sin(k_x x) + \hat{z}k_x\cos(k_x x))e^{-jk_z z}.
\end{aligned}$$

(b) Using the result of part (a), we have

$$\begin{aligned}
\eta_{TE} &= \frac{E_y}{-H_x} = \frac{2jE_o e^{-jk_z z}\sin(k_x x)}{\frac{2E_o}{\omega\mu_o}jk_z\sin(k_x x)e^{-jk_z z}} \\
&= \frac{1}{\frac{1}{\omega\mu_o}k_z} = \frac{\omega\mu_o}{\frac{\omega}{c}\sqrt{1 - \frac{f_c^2}{f^2}}} = \frac{\eta_o}{\sqrt{1 - \frac{f_c^2}{f^2}}}.
\end{aligned}$$