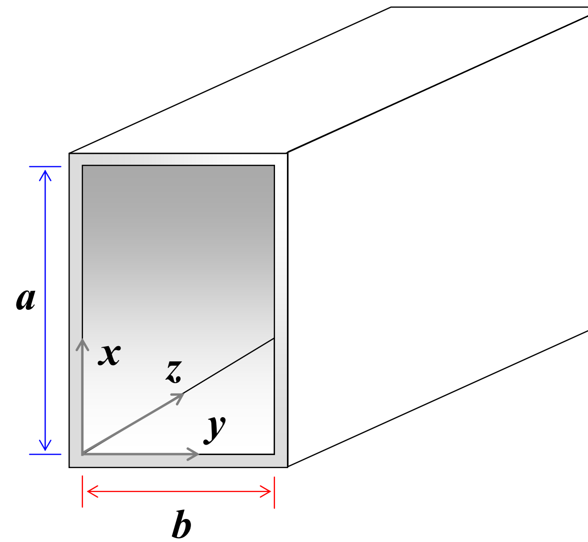
28 TM$_{mn}$ modes in rectangular waveguides



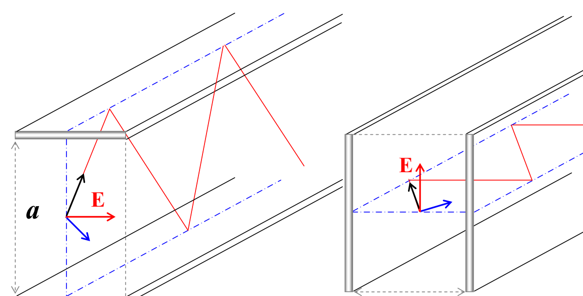
When the operation frequency f in a parallel-plate waveguide exceeds the cutoff frequency $f_c = \frac{c}{2a}$ of the TE$_1$ mode, dual- or multi-mode operations become unavoidable in the guide.

Single-mode operation at high frequencies can be attained by turning off the guided TEM(=TM$_0$) mode by introducing a pair of new plates on, say, $y = 0$ and $y = b$ planes as shown in the margin. This configuration is known as the "rectangular waveguide", which is the subject of the next set of lectures.

- Briefly, the guided TEM mode is suppressed in the rectangular waveguide, and propagation is only possible in terms of TM$_{mn}$ and TE$_{mn}$ modes. By definition:

  1. $H_z = 0$ for TM$_{mn}$ mode, for which the mode properties can be derived from a non-zero $E_z(x, y, z) = f(x, y)e^{-jk_z z}$;
  2. $E_z = 0$ for TE$_{mn}$ mode, for which the mode properties can be derived from a non-zero $H_z(x, y, z) = f(x, y)e^{-jk_z z}$;

  where the constraints on $f(x, y)$ and k_z are to be determined from Maxwell's equations and the relevant boundary conditions.



- Both TM$_{mn}$ and TE$_{mn}$ modes consist of the superposition of free-propagating TEM wave fields reflecting from the guide walls and satisfying the well-known vector wave equations

$$\nabla^2\tilde{\mathbf{E}} + \omega^2\mu_o\epsilon_o\tilde{\mathbf{E}} = 0 \quad \text{and} \quad \nabla^2\tilde{\mathbf{H}} + \omega^2\mu_o\epsilon_o\tilde{\mathbf{H}} = 0$$

derived from (see margin) Maxwell's equations.

Vector wave equation in phasor form: Taking the curl of Faraday's law

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu_o\tilde{\mathbf{H}},$$

and using

$$\begin{aligned} \nabla \times \nabla \times \tilde{\mathbf{E}} &= \nabla(\nabla \cdot \tilde{\mathbf{E}}) - \nabla^2\tilde{\mathbf{E}}, \\ \nabla \cdot \tilde{\mathbf{E}} &= 0, \\ \nabla \times \tilde{\mathbf{H}} &= j\omega\epsilon_o\tilde{\mathbf{E}}, \end{aligned}$$

it follows that

$$\nabla^2\tilde{\mathbf{E}} + \omega^2\mu_o\epsilon_o\tilde{\mathbf{E}} = 0.$$

Likewise,

$$\nabla^2\tilde{\mathbf{H}} + \omega^2\mu_o\epsilon_o\tilde{\mathbf{H}} = 0.$$

$\mathbf{TM}_{mn}$ modes:

- To examine the TM_{mn} mode with

$$H_z = 0 \quad \text{and} \quad E_z(x,y,z) = f(x,y)e^{-jk_z z}$$

consider the z-component of the wave-equation for the electric field, namely

$$\nabla^2 E_z + k^2 E_z = 0,$$

where

$$k^2 \equiv \omega^2\mu_o\epsilon_o \quad \text{and} \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Substituting E_z into the wave-equation component we have

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})f(x,y)e^{-jk_z z} + k^2 f(x,y)e^{-jk_z z} = 0,$$

from which

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})f(x,y)e^{-jk_z z} + (-jk_z)^2 f(x,y)e^{-jk_z z} + k^2 f(x,y)e^{-jk_z z} = 0$$

or

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})f(x,y) + (k^2 - k_z^2)f(x,y) = 0.$$

- We will next solve this 2D pdf using the method of **separation of variables**. In this method we assume that

$$f(x,y) = X(x)Y(y),$$

that is, we assume[1] that 2D function $f(x,y)$ of variables x and y is a product of 1D functions $X(x)$ and $Y(y)$ of x and y, respectively. With this assumption, the pdf above takes the form

$$YX'' + XY'' + (k^2 - k_z^2)XY = 0 \;\Rightarrow\; \frac{X''}{X} + \frac{Y''}{Y} + (k^2 - k_z^2) = 0$$

where

$$X'' \equiv \frac{\partial^2 X}{\partial x^2} \text{ and } Y'' \equiv \frac{\partial^2 Y}{\partial y^2}$$

  - Since $(k^2 - k_z^2)$ is independent x and y, it follows from the above pdf that X''/X as well as Y''/Y are constants independent of spatial coordinates. Thus we can write

$$\frac{X''}{X} = -k_x^2 \;\Rightarrow\; \frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0$$

where k_x is some constant. Also, by the same argument,

$$\frac{Y''}{Y} = -k_y^2 \;\Rightarrow\; \frac{\partial^2 Y}{\partial y^2} + k_y^2 Y = 0,$$

[1] This may appear to be a restricting assumption. However, if the procedure produces an infinite family of solutions (modes) which span the space of permissible solutions — i.e., a complete set in mathematical terms — then the procedure is not a restricting one in connection with linear pdf's that allow the superposition of permissible solutions.

$$(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})f+(k^2-k_z^2)f = 0$$

2.29 cm by 1.012 cm Stantard X-band (8.2-12.4 GHz) waveguide in which only TE_{10} mode is non-evanescent within X-band.

where k_y is some other constant. Furthermore, utilizing both of these conditions within

$$\frac{X''}{X} + \frac{Y''}{Y} + (k^2 - k_z^2) = 0$$

we get

$$-k_x^2 - k_y^2 + (k^2 - k_z^2) = 0 \quad \Rightarrow \quad k_z = \sqrt{k^2 - k_x^2 - k_y^2}.$$

$$\frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0$$

$$\frac{\partial^2 Y}{\partial y^2} + k_y^2 Y = 0,$$

- We continue by noting that the 2nd order ODEs for $X(x)$ and $Y(y)$ above are solved by

$$X(x) = A\cos k_x x + B \sin k_x x \quad \text{and} \quad Y(y) = C \cos k_y y + D \sin k_y y.$$

These general solutions with constants A, B, C, D simplify when we apply the boundary conditions that $X(x)Y(y) = 0$ at $x = 0$ and $y = 0$ as follows:

- $X(0) = 0$ implies $A = 0$, and in turn $X(x) = B \sin k_x x$;
- $Y(0) = 0$ implies $C = 0$, and in turn $Y(y) = D \sin k_y y$;

Furthermore,

- $X(a) = 0$ implies $k_x a = m\pi$, $m = 1, 2, 3, \cdots$
- $Y(b) = 0$ implies $k_y b = n\pi$, $n = 1, 2, 3, \cdots$

– Combining the above results, we get

$$f(x, y) = X(x)Y(y) = E_o \sin(k_x x) \sin(k_y y)$$

and consequently

$$E_z(x, y, z) = E_o \sin(k_x x) \sin(k_y y) e^{-jk_z z},$$

with

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad k_z = \frac{\omega}{c}\sqrt{1 - \frac{k_x^2 + k_y^2}{k^2}} = \frac{\omega}{c}\sqrt{1 - \frac{f_c^2}{f^2}},$$

where

$$f_c = \sqrt{(\frac{mc}{2a})^2 + (\frac{nc}{2b})^2}$$

is the pertinent cutoff frequency of the TM_{mn} mode with $m, n \geq 0$.

– Note that neither $m = 0$ nor $n = 0$ are permitted with non-zero E_z. Thus TM_{m0} and TM_{0n} modes don't exist!

Cutoff wavelength: As usual we have

$$\frac{\lambda_c}{\lambda} = \frac{f}{f_c}$$

and hence

$$\lambda_c = \frac{\lambda f}{\sqrt{(\frac{mc}{2a})^2 + (\frac{nc}{2b})^2}} = \frac{1}{\sqrt{(\frac{m}{2a})^2 + (\frac{n}{2b})^2}}.$$

Transverse field components:

Above, we have obtained the dispersion relation for TM_{mn} mode in rectangular waveguides. The dispersion characteristics of these modes are identical to those we have discussed in connection with parallel-plate waveguides except for the generalized expression for f_c.

- Given $E_z(x, y, z)$ determined above as well as the fact that $H_z = 0$ (by assumption), transverse field components of TM_{mn} mode waves can be inferred from Faraday's and Ampere's laws as shown next:

– With field components varying with z according to $e^{-jk_z z}$, Faraday's law implies

$$\nabla \times \tilde{\mathbf{E}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -jk_z \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu_o(H_x, H_y, H_z),$$

from which

$$H_x = \frac{\frac{\partial E_z}{\partial y} + jk_z E_y}{-j\omega\mu_o}, \quad H_y = \frac{\frac{\partial E_z}{\partial x} + jk_z E_x}{j\omega\mu_o}, \quad H_z = \frac{\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}}{-j\omega\mu_o}.$$

– Likewise, Ampere's law implies

$$\nabla \times \tilde{\mathbf{H}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -jk_z \\ H_x & H_y & H_z \end{vmatrix} = j\omega\epsilon_o(E_x, E_y, E_z),$$

from which

$$E_x = \frac{\frac{\partial H_z}{\partial y} + jk_z H_y}{j\omega\epsilon_o}, \quad E_y = \frac{\frac{\partial H_z}{\partial x} + jk_z H_x}{-j\omega\epsilon_o}, \quad E_z = \frac{\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}}{j\omega\epsilon_o}.$$

- Now (as confirmed in HW),

$$H_x = \frac{\frac{\partial E_z}{\partial y} + jk_z E_y}{-j\omega\mu_o} \text{ and } E_y = \frac{\frac{\partial H_z}{\partial x} + jk_z H_x}{-j\omega\epsilon_o}$$

from above imply that

$$H_x = -\frac{jk_z \frac{\partial H_z}{\partial x} - j\omega\epsilon_o \frac{\partial E_z}{\partial y}}{k^2 - k_z^2} \text{ and } E_y = -\frac{jk_z \frac{\partial E_z}{\partial y} - j\omega\mu_o \frac{\partial H_z}{\partial x}}{k^2 - k_z^2}$$

and, likewise,

$$H_y = \frac{\frac{\partial E_z}{\partial x} + jk_z E_x}{j\omega\mu_o} \text{ and } E_x = \frac{\frac{\partial H_z}{\partial y} + jk_z H_y}{j\omega\epsilon_o}$$

imply that

$$H_y = -\frac{jk_z \frac{\partial H_z}{\partial y} + j\omega\epsilon_o \frac{\partial E_z}{\partial x}}{k^2 - k_z^2} \text{ and } E_x = -\frac{jk_z \frac{\partial E_z}{\partial x} + j\omega\mu_o \frac{\partial H_z}{\partial y}}{k^2 - k_z^2}.$$

- The expressions above provide the transverse field components in terms of transverse derivatives of longitudinal components E_z and H_z.
  - By setting $H_z = 0$, they yield the transverse field components for TM_{mn} modes shown in the margin.

Also,

  - By setting $E_z = 0$, they yield the transverse field components for TE_{mn} modes also shown in the margin.

TM mode fields:

$$H_x = \frac{j\omega\epsilon_o \frac{\partial E_z}{\partial y}}{k^2 - k_z^2},$$

$$H_y = \frac{-j\omega\epsilon_o \frac{\partial E_z}{\partial x}}{k^2 - k_z^2},$$

$$E_x = \frac{-jk_z \frac{\partial E_z}{\partial x}}{k^2 - k_z^2},$$

$$E_y = \frac{-jk_z \frac{\partial E_z}{\partial y}}{k^2 - k_z^2}.$$

TE mode fields:

$$E_x = \frac{-j\omega\mu_o \frac{\partial H_z}{\partial y}}{k^2 - k_z^2},$$

$$E_y = \frac{j\omega\mu_o \frac{\partial H_z}{\partial x}}{k^2 - k_z^2},$$

$$H_x = \frac{-jk_z \frac{\partial H_z}{\partial x}}{k^2 - k_z^2},$$

$$H_y = \frac{-jk_z \frac{\partial H_z}{\partial y}}{k^2 - k_z^2}.$$