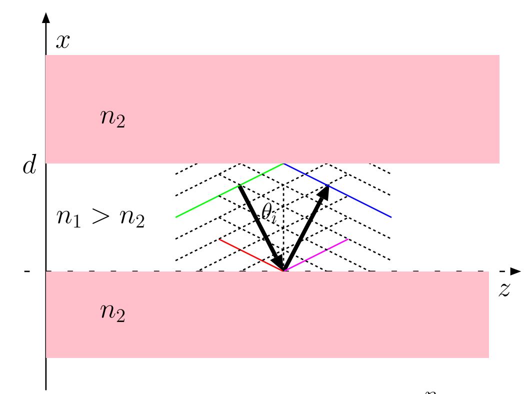
# 31 TE modes in dielectric slab waveguides

- As frequency $f$ increases well beyond the microwave range, the cutoff wavelength $\lambda_c = \frac{2a}{1} = \frac{c}{f}$ of the $TE_{10}$ of mode will dip towards $\mu$m scales. Guiding structures with $\mu$m scales can be more naturally implemented as dielectric slabs as opposed to hollow waveguides. Optical integrated circuits contain many such channels of **dielectric slab waveguides**.

  – In this lecture we will examine briefly the guidance conditions and dispersion characteristics encountered in dielectric slab waveguides.



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- Consider a slab of dielectric material of refractive index $n_1 = \sqrt{\epsilon_{1r}}$ of a width $d$ embedded in a dielectric with a smaller refractive index $n_2 = \sqrt{\epsilon_{2r}}$. Propagating modes of frequency $f$ can be trapped and guided in the slab with the refractive index $n_1 > n_2$,

  – so long as the mode can be represented as a superposition of unguided TEM waves reflected from plane boundaries of regions with index $n_1$ and $n_2$ with an incidence angle $\theta_i$ larger that $\theta_c$, where

$$\theta_c = \sin^{-1}\frac{n_2}{n_1}$$

  is the **critical angle** for **total internal reflection** (TIR).

  – Recall that when TIR occurs, the reflected wave has the same amplitude as the incident wave, while an evanescent transmitted wave is found in the second region. If $\theta_i < \theta_c$ no guidance can occur since the transmitted fields in that case would be propagating rather than evanescent.

- Guided modes not only require

$$\theta_i > \sin^{-1}\frac{n_2}{n_1},$$

but also

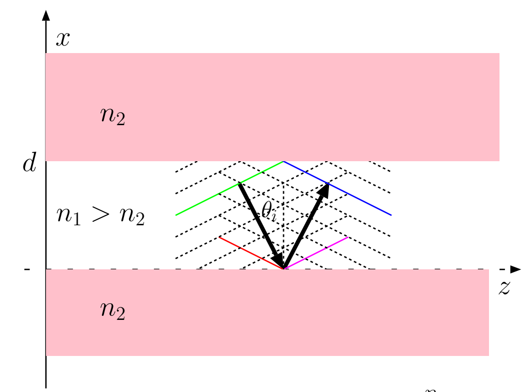
$$k_1 d\cos\theta_i = \angle\Gamma + m\pi, \quad m = 0, 1, 2, \cdots$$

where $\Gamma$ denotes the reflection coefficient at the interfaces between the regions of $n_1$ and $n_2$. This guidance condition ensures the self-consistency of free TEM components of the guided modes reflected from the planar interfaces separated by distance $d$.

$$\begin{aligned}\phi_b &= \phi_r - k_{1x}d,\\ \phi_g &= \phi_b + \angle\Gamma,\\ \phi_m &= \phi_g - k_{1x}d,\\ \phi_r &= \phi_m + \angle\Gamma\end{aligned}$$

where

$$k_{1x} = k_1\cos\theta_i.$$




Guidance requires $\theta_i > \theta_c = \sin^{-1}\frac{n_2}{n_1}$

$$k_1 d\cos\theta_i = \angle\Gamma + m\pi, \quad m = 0, 1, 2, \cdots$$

- For the TE mode case where the incident and reflected fields taken as

$$\tilde{\mathbf{E}}_i = \hat{y}E_o e^{-jk_1(-\cos\theta_i x + \sin\theta_i z)} \text{ and } \tilde{\mathbf{E}}_r = \hat{y}E_o\Gamma_{TE}e^{-jk_1(\cos\theta_i x + \sin\theta_i z)}$$

the reflection coefficient is given as

$$\Gamma_{TE} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_2}{\eta_2\cos\theta_i + \eta_1\cos\theta_2} = \frac{n_1\cos\theta_i - n_2\cos\theta_2}{n_1\cos\theta_i + n_2\cos\theta_2}$$

since

$$\eta = \sqrt{\frac{\mu_o}{\epsilon_r\epsilon_o}} = \frac{\eta_o}{\sqrt{\epsilon_r}} = \frac{\eta_o}{n}.$$

- Above,

$$\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_i}$$

since according to Snell's law

$$k_1\sin\theta_i = k_2\sin\theta_2 \text{ and } \sin\theta_2 = \frac{k_1}{k_2}\sin\theta_1 = \frac{\omega\sqrt{\mu_o\epsilon_1}}{\omega\sqrt{\mu_o\epsilon_2}}\sin\theta_1 = \frac{n_1}{n_2}\sin\theta_1.$$

Clearly, for

$$1 < \frac{n_1^2}{n_2^2}\sin^2\theta_i \quad\Leftrightarrow\quad \theta_i > \theta_c = \sin^{-1}\frac{n_2}{n_1}$$

we have

$$\cos\theta_2 = \pm j\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1}$$

and (using the root that causes the decay of the fields $\propto e^{\mp jk_2\cos\theta_2 x}$ above and below the slab)

$$\Gamma_{TE} = \frac{n_1\cos\theta_i + j\sqrt{n_1^2\sin^2\theta_i - n_2^2}}{n_1\cos\theta_i - j\sqrt{n_1^2\sin^2\theta_i - n_2^2}}$$

with

$$\angle\Gamma_{TE} = 2\tan^{-1}\frac{\sqrt{\sin^2\theta_i - n_2^2/n_1^2}}{\cos\theta_i}.$$

Now, substituting $\angle\Gamma_{TE}$ in the guidance condition shown in the margin, we obtain

$$\frac{k_1 d}{2}\cos\theta_i - m\frac{\pi}{2} = \tan^{-1}\frac{\sqrt{\sin^2\theta_i - n_2^2/n_1^2}}{\cos\theta_i}, \quad m = 0, 1, 2, 3, \cdots$$

**Guidance conditions:**

$$\theta_i > \sin^{-1}\frac{n_2}{n_1}$$

and

$$k_1 d\cos\theta_i = \angle\Gamma + m\pi, \quad m = 0, 1, 2, \cdots$$

**Another way to obtain the guidance conditions:**

Since

$$\tilde{\mathbf{E}}_i = \hat{y}E_o e^{-jk_1(-\cos\theta_i x + \sin\theta_i z)}$$

and

$$\tilde{\mathbf{E}}_r = \hat{y}E_o\Gamma_{TE} e^{-jk_1(\cos\theta_i x + \sin\theta_i z)},$$

and $\tilde{\mathbf{E}}_r$ gets reflected at $x = d$ (once again) to become $\tilde{\mathbf{E}}_i$, it is then necessary that

$$(\Gamma_{TE} e^{-jk_1\cos\theta_i d})\Gamma_{TE} = e^{jk_1\cos\theta_i d}e^{-j2\pi m}$$

for integers $m$. This is possible iff $|\Gamma_{TE}| = 1$, i.e.,

$$\theta_i > \sin^{-1}\frac{n_2}{n_1},$$

and

$$k_1 d\cos\theta_i = \angle\Gamma_{TE} + m\pi, \quad m = 0, 1, 2, \cdots$$

which is only valid for $\theta_i$ satisfying

$$\theta_i \geq \sin^{-1}\frac{n_2}{n_1} \equiv \theta_c.$$

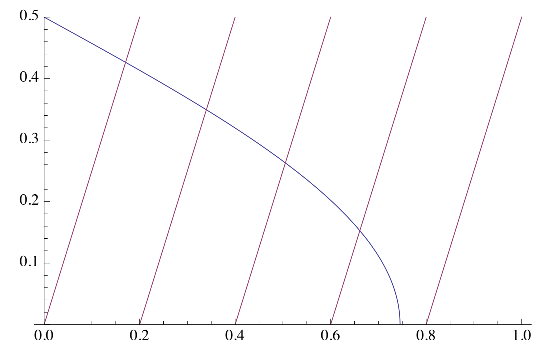
Since

$$k_1 = \frac{\omega}{v_1}, \quad \text{where } v_1 = \frac{1}{\sqrt{\mu_o \epsilon_1}} = \frac{c}{n_1},$$

the guidance condition can also be cast as

$$\frac{d}{v_1/f}\cos\theta_i - \frac{m}{2} = \frac{1}{\pi}\tan^{-1}\frac{\sqrt{\sin^2\theta_i - n_2^2/n_1^2}}{\cos\theta_i}, \quad m = 0, 1, 2, 3, \cdots$$

which is known as the **characteristic equation** for TE modes.




Graphical solution of the characteristic equation for TE modes $m = 0, 1, 2, 3$ in propagation and mode $m = 4$ in evanescence. The blue curve depicts

$$\frac{1}{\pi}\tan^{-1}\frac{\sqrt{\sin^2\theta_i - n_2^2/n_1^2}}{\cos\theta_i}$$

as a function of $\cos\theta_i$ for $n_2 = 1$ and $n_1 = 1.5$ while the straight lines depict

$$\frac{d}{v_1/f}\cos\theta_i - \frac{m}{2}$$

with $\frac{d}{v_1/f} = 2.5$ and $m$ increasing from left to right in steps of one.

- The above equations constrain the number of propagating modes at a given frequency $f$ and the associated angle $\theta_i$ for each TE mode $m$.
  - Each propagating mode for a given $d$ is associated with a cutoff frequency $f_c$, and propagation is possible only if $f > f_c$ for the given mode.
  - At $f = f_c$ we have $\theta_i = \theta_c$ for the given mode, in which case

$$\frac{d}{v_1/f_c}\cos\theta_c - \frac{m}{2} = \frac{1}{\pi}\tan^{-1}\frac{\sqrt{\sin^2\theta_c - n_2^2/n_1^2}}{\cos\theta_c} = 0,$$

from which we obtain the cutoff frequencies

$$f_c = \frac{m v_1}{2d\cos\theta_c}, \quad m = 0, 1, 2, 3\cdots$$

Since

$$v_1 = \frac{c}{n_1} \quad \text{and} \quad \cos\theta_c = \sqrt{1-\sin^2\theta_c} = \sqrt{1-\frac{n_2^2}{n_1^2}}$$

it follows that

$$f_c = \frac{mc}{2d\sqrt{n_1^2-n_2^2}}, \quad m = 0, 1, 2, 3\cdots$$

for TE modes.

**Example 1:** Consider a dielectric slab waveguide with $d = 3$ mm, $n_1 = 1.5$, and $n_2 = 1$. (a) Determine the cutoff frequency for the $TE_1$ mode in the guide. (b) Determine the frequency $f$ of a $TE_0$ mode signal in the waveguide if $\theta_i = 60°$. (c) Determine the phase velocity of the mode described in part (b).

**Solution:** (a) The cutoff frequency for $TE_1$ mode is

$$f_c = \frac{mc}{2d\sqrt{n_1^2-n_2^2}} = \frac{3\times 10^{10}\,\text{cm/s}}{2\times 0.3\,\text{cm}\sqrt{1.5^2-1}} = \frac{5\times 10^{10}}{\sqrt{1.25}}\,\text{Hz} \approx 44.72\,\text{GHz}.$$

(b) Evaluating the characteristic equation

$$\frac{d}{v_1/f}\cos\theta_i - \frac{m}{2} = \frac{1}{\pi}\tan^{-1}\frac{\sqrt{\sin^2\theta_i - n_2^2/n_1^2}}{\cos\theta_i}, \quad m = 0, 1, 2, 3, \cdots$$

with $m = 0$, $n_2/n_1 = 2/3$, $\sin\theta_i = \sqrt{3}/2$, and $\cos\theta_i = 1/2$, we find

$$\frac{fd}{v_1 2} = \frac{1}{\pi}\tan^{-1}\frac{\sqrt{3/4-(2/3)^2}}{1/2} = 0.266 \quad \Rightarrow \quad f = 0.266\times 2\times\frac{v_1}{d}.$$

Since

$$v_1 = \frac{c}{n_1} = \frac{3 \times 10^{10}\,\mathrm{cm/s}}{3/2} = 2 \times 10^{10}\,\mathrm{cm/s},$$

we find

$$f = 0.266 \times 2 \times \frac{v_1}{d} = 0.266 \times 2 \times \frac{2 \times 10^{10}\,\mathrm{cm/s}}{0.3\,\mathrm{cm}} = 3.54 \times 10^{10}\,\mathrm{Hz} = 35.4\,\mathrm{GHz}.$$

(c) The phase velocity of the mode is given by

$$v_{pz} = \frac{\omega}{k_z} = \frac{\omega}{k_1 \sin\theta_i} = \frac{v_1}{\sin\theta_i} = \frac{2 \times 10^{10}\,\mathrm{cm/s}}{\sqrt{3}/2} = 2.31 \times 10^{10}\,\mathrm{cm/s}.$$