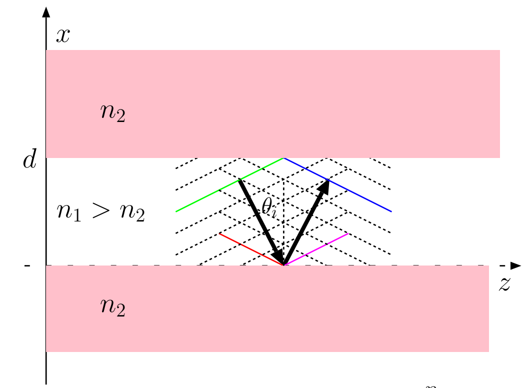
32 TM modes in dielectric waveguides



Guidance requires $\theta_i > \theta_c = \sin^{-1}\frac{n_2}{n_1}$

- Last lecture we examined the **characteristic equation** and the **cutoff frequencies** of TE mode of propagation in dielectric slab waveguides.

  Guided TE$_m$ mode fields consisting of the superposition of transverse polarized electric fields

  $$\tilde{\mathbf{E}}_i = \hat{y}E_o e^{-jk_1(-\cos\theta_i x + \sin\theta_i z)} \text{ and } \tilde{\mathbf{E}}_r = \hat{y}E_o\Gamma_{TE}e^{-jk_1(\cos\theta_i x + \sin\theta_i z)},$$

  where

  $$\Gamma_{TE} = \frac{n_1\cos\theta_i + j\sqrt{n_1^2\sin^2\theta_i - n_2^2}}{n_1\cos\theta_i - j\sqrt{n_1^2\sin^2\theta_i - n_2^2}},$$

  have

  1. propagation angles

  $$\theta_i > \theta_c = \sin^{-1}\frac{n_2}{n_1}, \quad \text{(critical angle)}$$

  2. satisfying a characteristic equation

  $$\frac{d}{v_1/f}\cos\theta_i - \frac{m}{2} = \frac{1}{\pi}\tan^{-1}\frac{\sqrt{\sin^2\theta_i - n_2^2/n_1^2}}{\cos\theta_i}, \quad m = 0, 1, 2, 3, \cdots$$

  3. for frequencies f exceeding the cutoff frequency

  $$f_c = \frac{mc}{2d\sqrt{n_1^2 - n_2^2}}, \quad m = 0, 1, 2, 3\cdots$$

- For a given $f = \frac{2\pi}{\omega}$, the charcteristic equation can be solved (typically by using graphical techniques) for θ_i, from which we can calculate the propagation constant

$$k_z = k_1 \sin\theta_i$$

where

$$k_1 = \frac{\omega}{v_1} = \frac{\omega}{c} n_1$$

is the wavenumber in the core region of the guide at the operation frequency $\omega = 2\pi f$. It follows that the

$$\text{guide wavelength } \lambda_g = \frac{2\pi}{k_z}$$

and [1]

$$\text{phase velocity } v_{pz} = \frac{\omega}{k_z} = \frac{\omega}{k_1 \sin\theta_i} = \frac{v_1}{\sin\theta_i}$$

can be obtained once θ_i is calculated from the characteristic equation.

- Given the k_z in the core region, k_x and k_z outside the core region (with index n_2) can be obtained by using the fact that k_z is identical in both regions (why?).

- In this lecture we will continue our study of dielectric slab waveguides by examining the TM modes.

[1]Note that group velocity $v_g = v_1 \sin\theta_i$ (in analogy with parallel plate wavegiudes) if and only if $\omega \gg \omega_c = 2\pi f_c$ because of the effect of the cladding region that contains a substatial fraction on the wave energy unless $\omega \gg \omega_c$ — in fact for TE_0 mode $v_g \approx v_2$ at frequencies much less than the cutoff frequency of TE_1 mode.

TM modes

- For the TM mode where the incident and reflected fields are taken as

$$\tilde{\mathbf{H}}_i = \hat{y} H_o e^{-jk_1(-\cos\theta_i x + \sin\theta_i z)} \text{ and } \tilde{\mathbf{H}}_r = \hat{y} H_o R e^{-jk_1(\cos\theta_i x + \sin\theta_i z)}$$

the reflection coefficient is given as

$$R = \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_2}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_2} = \frac{n_2 \cos\theta_i - n_1 \cos\theta_2}{n_2 \cos\theta_i + n_1 \cos\theta_2}.$$

This leads to

$$R = \frac{n_2 \cos\theta_i + j n_1 \sqrt{\frac{n_1^2}{n_2^2} \sin^2\theta_i - 1}}{n_2 \cos\theta_i - j n_1 \sqrt{\frac{n_1^2}{n_2^2} \sin^2\theta_i - 1}}$$

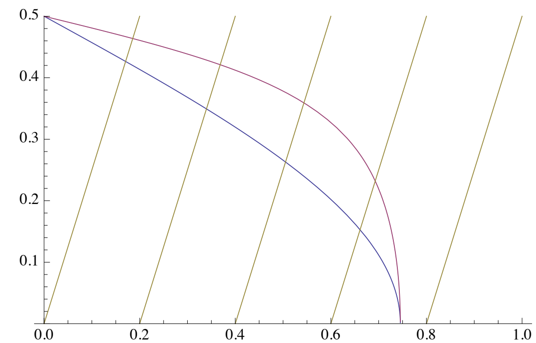
with

$$\angle R = 2 \tan^{-1} \frac{\sqrt{\sin^2\theta_i - n_2^2/n_1^2}}{\frac{n_2^2}{n_1^2} \cos\theta_i}.$$

Hence, in this case the guidance condition leads to the characteristic equation

$$\frac{d}{v_1/f} \cos\theta_i - \frac{m}{2} = \frac{1}{\pi} \tan^{-1} \frac{\sqrt{\sin^2\theta_i - n_2^2/n_1^2}}{\frac{n_2^2}{n_1^2} \cos\theta_i}, \quad m = 0, 1, 2, 3, \cdots$$

Note that this result for the TM mode leads to the same f_c expression as in TE modes.



Graphical solution of the characteristic equations for TE and TM modes $m = 0, 1, 2, 3$ in propagation and mode $m = 4$ in evanescence. The blue and red curves depict

$$\frac{1}{\pi} \tan^{-1} \frac{\sqrt{\sin^2\theta_i - n_2^2/n_1^2}}{\cos\theta_i}$$

and

$$\frac{1}{\pi} \tan^{-1} \frac{\sqrt{\sin^2\theta_i - n_2^2/n_1^2}}{\frac{n_2^2}{n_1^2} \cos\theta_i}$$

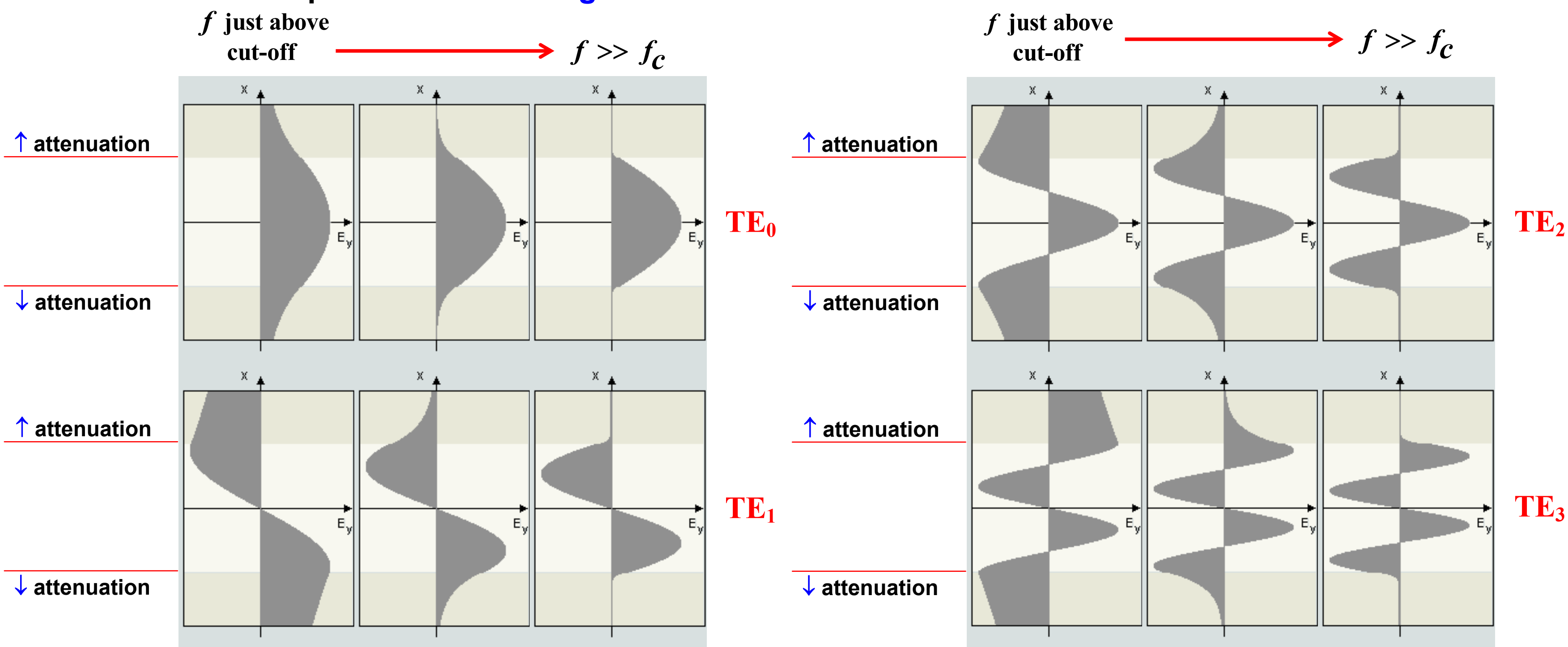
as a function of $\cos\theta_i$ for $n_2 = 1$ and $n_1 = 1.5$, while the straight lines depict

$$\frac{d}{v_1/f} \cos\theta_i - \frac{m}{2}$$

with $\frac{d}{v_1/f} = 2.5$ and m increasing from left to right in steps of one.

Mode structures:

Examples of profiles for the transverse electric field of TE modes.
TM modes have similar profiles for the magnetic field.



Acceptance cone and numerical aperture:

Guidance requires

$$\theta_i > \theta_c = \sin^{-1}\frac{n_2}{n_1} = \sin^{-1}\sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}$$

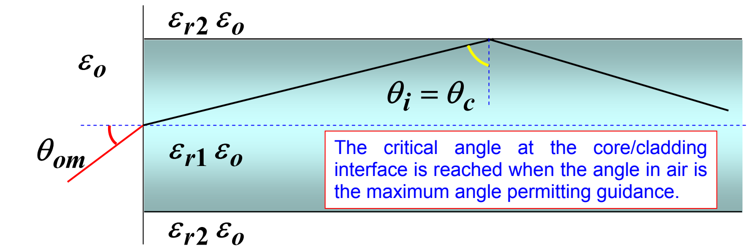
and therefore "acceptence angles" from air (see the diagram below)

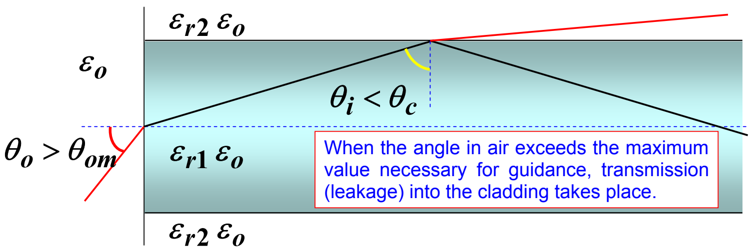
$$\theta_o < \theta_{om} = \sin^{-1}\sqrt{n_1^2 - n_2^2} = \sin^{-1}\sqrt{\epsilon_{r1} - \epsilon_{r2}}$$

where

$$\sin\theta_{om} = \sqrt{n_1^2 - n_2^2} = \sqrt{\epsilon_{r1} - \epsilon_{r2}}$$ is called **numerical aperture**.

The maximum acceptance angle θ_{om} defines the so-called "acceptance cone" that includes all the external signals incident on the dielectric waveguide that can couple to the waveguide at the air/core interface on a constant-z plane.

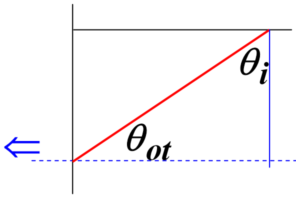




At the air–core interface

$$\sin\theta_{ot} = \sqrt{\frac{\varepsilon_{rair}\varepsilon_o}{\varepsilon_{r1}\varepsilon_o}}\sin\theta_o = \sqrt{\frac{1}{\varepsilon_{r1}}}\sin\theta_o$$

$$\theta_i + \theta_{ot} = 90° \quad \Rightarrow \quad \cos\theta_{ot} = \sin\theta_i \quad \Leftarrow$$



At the critical angle

$$\sin\theta_i = \sin\theta_c = \sqrt{\frac{\varepsilon_{r2}}{\varepsilon_{r1}}}$$

$$\sin^2\theta_{otm} = 1 - \cos^2\theta_{otm} = 1 - \sin^2\theta_c = 1 - \frac{\varepsilon_{r2}}{\varepsilon_{r1}} = \frac{\sin^2\theta_{om}}{\varepsilon_{r1}}$$

$$\Rightarrow \quad \sin\theta_{om} = \sqrt{\varepsilon_{r1} - \varepsilon_{r2}} = \textbf{numerical aperture}$$

$$\theta_{om} = \sin^{-1}\sqrt{\varepsilon_{r1} - \varepsilon_{r2}}$$