34 Resonant modes and field fluctuations

- Since in a rectangular cavity the resonant frequencies

$$f_{mnl} = \frac{c}{2}\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2 + (\frac{l}{d})^2} \;\Rightarrow\; \frac{2f_{mnl}}{c} = \sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2 + (\frac{l}{d})^2},$$

we can consider $2f_{mnl}/c$ to be the "length" of a "vector" $(\frac{m}{a}, \frac{n}{b}, \frac{l}{d})$ pointing away from the origin of a "3D Cartesian space" where each *lattice point*, e.g., $(\frac{m}{a}, \frac{n}{b}, \frac{l}{d}) = (\frac{1}{a}, \frac{2}{b}, \frac{1}{d})$, is associated with two resonant modes (TE and TM) of the cavity.

  - In this space, "volume" per lattice point is $\frac{1}{abd}$, and thus *volume per resonant mode* is $\frac{1/2}{abd}$.
  - Also, all the resonant modes with resonance frequencies f_{mnl} less than a given frequency f can be associated with lattice points residing within one eight (an octant) of a sphere of "radius" $2f/c$ centered about the origin of the same space — only an octant is involved since the indices m, n, l employed are all non-negative.

Thus, the number of resonant modes with frequencies less than f, to be denoted as the cumulative distribution $C(f)$, is found to be

$$C(f) = \frac{\frac{1}{8} \times (\text{sphere of radius } 2f/c)}{\frac{1/2}{abd}} = \frac{\frac{1}{8} \times \frac{4\pi}{3}(\frac{2f}{c})^3}{\frac{1/2}{abd}} = \frac{8\pi f^3}{3c^3}V$$

where $V = abd$ is the physical volume of the cavity. Consequently, the number density $N(f)$ of the available resonant modes in a cavity of volume V is obtained as

$$N(f) = \frac{dC}{df} = \frac{8\pi f^2}{c^3} V \quad \frac{\text{modes}}{\text{Hz}}$$

which grows quadratically with frequency f. As illustrated later in this lecture, the distribution $N(f)$ has deep theoretical implications.

Example 1: Consider a rectangular cavity with dimensions $a = b = d = 0.3$ m. Determine $N(f)$ for $f = 50$ GHz and the number of resonant modes to be found within a bandwidth of $\Delta f = 1$ GHz centered about $f = 50$ GHz.

Solution: Using the density function derived above, we find that

$$N(50 \times 10^9) = \frac{8\pi(50 \times 10^9)^2}{(3 \times 10^8)^3}(3 \times 10^{-1})^3 = 8\pi \times 25 \times 10^{-7} = 2\pi \times 10^{-5} \, \frac{\text{modes}}{\text{Hz}}.$$

Thus, the number of resonant modes in a bandwidth of $\Delta f = 1$ GHz centered about $f = 50$ GHz is

$$\text{Number of modes within band} = (2\pi \times 10^{-5} \, \frac{\text{modes}}{\text{Hz}}) \times 10^9 \, \text{Hz} = 20000\pi \approx 60000.$$

Energy spectrum of radiation in enclosed cavities:

- Consider an air-filled rectangular cavity with slightly lossy walls sitting on a table top in some lab where the room temperature is 300 K. Assume that the cavity has been in the room for a long time and has reached thermal equilibrium with the rest of the room — i.e., the walls of the cavity also have $T = 300$ K.

  It turns out that such a cavity will be filled with electromagnetic fields consisting of (i.e., a superposition of) the TE_{mnl} and TM_{mnl} modes distributed across the frequency space with a density function

  $$N(f) = \frac{8\pi f^2}{c^3} V$$

  derived above.

  – The resonant modes with the distribution function just quoted are the result of radiation by random currents flowing on the cavity walls caused by random thermal agitations of the charge carriers located within the walls.

  – As soon as it is (randomly) established, a resonant mode will start decaying because of ohmic losses in cavity walls (see earlier discussions), returning back the radiated energy of the wall back to the wall.

  – In thermal equilibrium the temperature of the wall as well as the *expected* total energy of cavity radiation summed over all of its modes will remain constant.

- The energy density spectrum of the radiation within the cavity, $E(f)$, measured in units of J/m^3/Hz, should be product of $N(f)/V$ with $\langle W(f)\rangle$ representing the expected value (statistical average) of the energy $W(f)$ of each mode at resonant frequency f. Hence,

$$E(f) = \frac{8\pi f^2}{c^3}\langle W(f)\rangle.$$

- What might be the expected mode energy $\langle W(f)\rangle$?

- Each resonant mode such as TE_{101} or TM_{112} can be interpreted as two *degrees of freedom* (one degree for **E** and one for **H**) of the electromagnetic field variations in a closed cavity, just as velocity components v_x, v_y, v_z of any one of N molecules contained within a volume of gas are each considered a "degree of freedom" for the N molecule system.

  - In physical models each degree of freedom in a gas in thermal equilibrium is assigned[1] an expected energy of

    $$\langle \frac{1}{2} m v_x^2 \rangle = \frac{1}{2} KT$$

    where $K \equiv 1.38 \times 10^{-23}$ J/K is *Boltzmann's constant* and T is the equilibrium temperature in K.

[1] In thermal equilibrium all particles have by definition equal average energies. Denoting this energy as $\frac{1}{2}KT$ is just a matter of *defining* the **equilibrium temperature** of the gas in terms the **average kinetic energy** of its individual molecules — at a fundamental level that *is* what temperature is! Including the Boltzmann constant K in this assignment is just a matter of setting the scale used for temperature (Kelvin scale by convention). At room temperature (298 K), KT works out to be 0.0256 eV.

- If we naively make a similar assignment (see margin note) to $\langle W(f)\rangle$, e.g., take

$$\langle W(f)\rangle = KT$$

(on account of the fact that TE_{mnl} and TM_{mnl} modes have energies which are the sum of two quadratic terms proportional to $|\tilde{\mathbf{E}}|^2$ and $|\tilde{\mathbf{H}}|^2$), we then immediately run into a difficulty in that $E(f)$ blows up to infinity in the high frequency end because $\langle W(f)\rangle$ has no high-frequency cutoff.

- The difficulty just mentioned — known as "ultraviolet catastrophe" — was well recognized at the beginning of the 20th century, and was resolved by Max Planck's recognition that electromagnetic mode energies $W(f_{mnl})$ have to be *quantized* in chunks of size hf_{mnl}, and

$$\langle W(f_{mnl})\rangle = KT$$

is acceptable only if an "energy quantum" $hf_{nml} \ll KT$.

If $hf_{nml} \gg KT$ for a given mode, then the mode is very seldom excited (to an energy level of one hf_{nml}), and thus the expected value of energy $W(f_{mnl})$ in the mode is an exponentially reduced fraction of an energy quantum hf_{nml} given by

$$\langle W(f_{mnl})\rangle = hf_{mnl}\, e^{-hf_{mnl}/KT}.$$

This effective "cutoff" in $\langle W(f)\rangle$ function eliminates the ultraviolet catastrophe.

$\frac{1}{2}KT$ per quadratic term: Assigning an expected energy of $\frac{1}{2}KT$ per quadratic term in a total energy expression of a large system of elements in thermal equilibrium is a standard procedure used in classical statistical mechanics. This is a consequence of well known experimental results such as: in a gas consisting of a mixture of light and heavy atoms, $\langle\frac{1}{2}mV_x^2\rangle$ of the light atoms match $\langle\frac{1}{2}Mv_x^2\rangle$ of the heavy atoms in thermal equilibrium — all quadratic energy terms get the same $\frac{1}{2}KT$ (clasically)!

– Using the **1st** and **2nd laws of thermodynamics** together with the **quantization rule** that he introduced, Planck derived[2] the relation

$$\langle W(f_{mnl})\rangle = \frac{hf_{mnl}}{e^{hf_{mnl}/KT} - 1}$$

for the expected mode energies having the limiting cases for

$$hf_{nml} \ll KT \text{ and } hf_{nml} \gg KT$$

just discussed.

– With this result, the energy spectrum within a cavity in thermal equilibrium takes the form

$$E(f) = \frac{N(f)}{V}\langle W(f)\rangle = \frac{8\pi f^2}{c^3}\frac{hf}{e^{hf/KT} - 1}\frac{\mathrm{J/m^3}}{\mathrm{Hz}}.$$

This derived spectral shape was successfully adjusted to fit the observed energy spectra of cavity radiation by varying the parameter h, which is now known as *Planck's constant*[3] and has the fixed value of 6.626×10^{-34} Js.

Shape independence: $E(f)$ obtained for the rectangular cavity is actually *independent* of cavity shape. This can be justified by considering two cavities, one rectangular, one not, joined by a small aperture. If the two cavities have the same temperature T, then *by definition* (of T) there cannot be any net energy exchange between the cavities at any f (a *detailed balance* per frequency is required because the aperture may have an f dependent transmittivity) — hence a common $E(f)$ for the two cavities with a common T even if the shapes are different!

$E(f)$ is also independent of the lossiness of the walls (even though Q of the cavity depends on it) and therefore applicable to *all* lossy cavities at thermal equilibrium including those whose walls are *perfect absorbers*, i.e., **blackbodies**.

[2]See *Oliver, B. M.*, "Thermal and Quantum Noise", Proc IEEE, **53**, 436 (1965) for a simplified version of Planck's derivation.

[3]Planck's constant h is one of the three fundamental constants of physics, along with c and G, the gravitational constant, from which *absolute units* for all physical variables can be derived in suitable combinations: e.g., length unit=$\sqrt{hG/c^3}$, time unit=$\sqrt{hG/c^5}$, etc.