35 Cavity radiation and thermal noise

- In a 1D cavity of some length L — e.g. a TL with *shorts* at both ends as discussed in ECE 329 notes — the resonant frequencies are

$$f_m = \frac{c}{\lambda_m} = \frac{c}{2L/m} = \frac{c}{2L}m, \quad \text{where } m = 1, 2, 3, \cdots$$

which indicates that the mode density in f is

$$N(f) = \frac{2L}{c} \ \frac{\text{modes}}{\text{Hz}}.$$

Therefore the energy density in a 1D cavity in thermal equilibrium (assume vanishingly lossy wires with temperature T) will be

$$E(f) = \frac{N(f)}{L}\langle W(f)\rangle = \frac{2}{c}\frac{hf}{e^{hf/KT}-1}\ \frac{\text{J/m}}{\text{Hz}}$$

in analogy with the energy density of 3D cavities. This energy density will reside by equal amounts in the traveling wave components of the 1D resonant modes arriving with speed c from the opposite ends of the 1D resonator. Power spectral content $P(f)$ of each of these traveling wave components can thus be calculated as $c/2$ times[1] $E(f)$, i.e.,

$$P(f) = \frac{hf}{e^{hf/KT}-1}\ \frac{\text{W}}{\text{Hz}}.$$

[1]Note that per TEM plane wave,

$$c(\frac{1}{4}\epsilon_o|\tilde{\mathbf{E}}|^2 + \frac{1}{4}\mu_o|\tilde{\mathbf{H}}|^2) = \frac{|\tilde{\mathbf{E}}|^2}{4\eta_o} + \eta_o\frac{|\tilde{\mathbf{H}}|^2}{4} = \frac{|\tilde{\mathbf{E}}|^2}{2\eta_o},$$

which confirms that the time-averaged stored energy density times c is indeed the time-averaged power transported per unit area.

Cavity radiance: Energy density

$$E(f) = \frac{N(f)}{V}\langle W(f)\rangle = \frac{8\pi f^2}{c^3}\frac{hf}{e^{hf/KT}-1}\ \frac{\text{J/m}^3}{\text{Hz}}.$$

in a 3D cavity in thermal equilibrium resides by equal amounts in the traveling wave components of the cavity modes arriving with speed c from the boundaries of the cavity subtending 4π sterads. Multiplying $E(f)$ by $c/4\pi$ we obtain

$$L(f) = \frac{2f^2}{c^2}\frac{hf}{e^{hf/KT}-1}\ \frac{\text{W/m}^2\text{/ster}}{\text{Hz}},$$

which is called *radiance* and represents the *power density per unit solid angle* of the waves traveling within the cavity.

Radiance $L(f)$ also represents the spectrum of power *radiated* per unit solid angle by a unit area of a **blackbody surface** at temperature T (since non-reflective walls of a cavity will produce the same $E(f)$ as partial-reflecting walls as mentioned earlier).

- Now replace the shorts at the ends of the resonator with resistors R at temperature T *matching* the characteristic impedance Z_o of the line.

  Since there cannot be any net power exchange between elements in thermal equilibrium (over *any* frequency band — otherwise a net broadband exchange can be arranged for by using filters with suitable frequency responses in violation of the 2nd law of thermodynamics), it follows that matched resistors R will be both absorbing (a full absorption because of impedance matching) and injecting (to a matched load again because of the same fact) the same power density $P(f)$ identified above.

- The upshot is, we need to conclude that any resistor R at a temperature T must have an *available* power density of

$$P(f) = \frac{hf}{e^{hf/KT} - 1} \frac{\text{W}}{\text{Hz}}$$

fueled by thermal agitation of its internal charge carriers. The resistor is then capable of outputting an average power of

$$P(f)\Delta f = \frac{hf}{e^{hf/KT} - 1}\Delta f \equiv \frac{\langle v^2 \rangle}{4R}$$

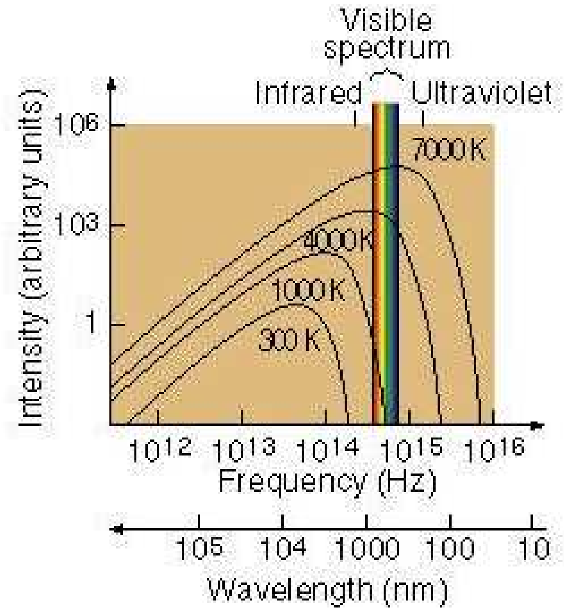
over a bandwidth Δf, where $\langle v^2 \rangle$ is the mean squared *open-circuit voltage* at the resistor terminals over the same bandwidth.

Unmatched termination case: If $R \neq Z_o$, then a portion $P(f)|\Gamma|^2$ of the incident power $P(f)$ will be reflected from R (rather than being fully absorbed). In that case R emits only a reduced level of power $P(f)(1 - |\Gamma|^2)$. This is a simple example of how $P(f)$ can be emitted in its entirety only by **perfect absorbers** defined to be blackbodies. In 1D, the *blackbody* radiance at temperature T is

$$P(f) = \frac{hf}{e^{hf/KT} - 1} \frac{\text{W}}{\text{Hz}}$$

while in 3D it is

$$L(f) = \frac{2f^2}{c^2} \frac{hf}{e^{hf/KT} - 1} \frac{\text{W/m}^2\text{/ster}}{\text{Hz}}.$$



- It follows that

$$\frac{\langle v^2\rangle}{4R} = \frac{hf\Delta f}{e^{hf/KT}-1} \text{ reducing to } \frac{\langle v^2\rangle}{4R} = KT\Delta f \text{ for } hf \ll KT,$$

  a result known as **Nyquist noise theorem**. The theorem can also be cast as

$$\frac{\langle i^2\rangle}{4G} = \frac{hf\Delta f}{e^{hf/KT}-1} \text{ reducing to } \frac{\langle i^2\rangle}{4G} = KT\Delta f \text{ for } hf \ll KT$$

  in terms of mean squared *short-circuit current* $\langle i^2\rangle$ of the same resistor over the same bandwidth and conductance $G = 1/R$. Note that if the element has an impedance $Z = R + jX = 1/Y$ only the real part of Z should be utilized in connection with power transferred to a matched load Z^*.

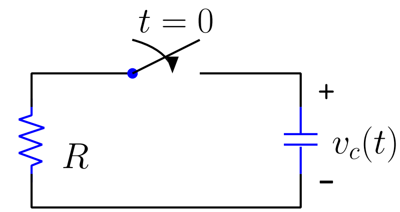
- Nyquist noise theorem outlined above has a very powerful generalization known as the **fluctuation-dissipation theorem**:

  – according to this theorem, any linear and dissipative system in thermodynamic equilibrium will exhibit thermally driven fluctuations of its dynamic parameters (e.g., electron density in plasma at finite temperature), and

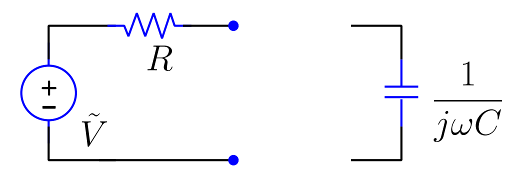
  – the frequency spectrum of the fluctuations can be obtained by applying the Nyquist noise theorem to an appropriately constructed equivalent circuit model of the system.

- The unavoidable fact of fluctuations and noise encountered in dissipative systems and circuits constitutes both a challenge and an opportunity for the engineer. Consider taking ECE 453 to develop a better understanding of noise issues in communication circuits.

(a) Initial value problem:



(b) Frequency domain Thevenin model of a noisy resistor:



Example: Consider the RC circuit shown in the margin. Assuming that the capacitor holds 10 V prior the switch is closed at $t = 0$, the capacitor voltage for $t > 0$ can be expressed as

$$v_c(t) = 10e^{-t/RC}$$

using ECE 210 knowledge. This solution implies the dissipation of the initial stored energy within the resistor. But as we have seen in this lecture, dissipative elements such as resistors also produce random thermal voltages and currents. We therefore expect a *non-zero* $v_c(t)$ in the circuit shown in the margin as $t \to \infty$, assuming that the resistor has some non-zero steady-state temperature T measured in Kelvins. Given that the resistor produces an open circuit voltage $v(t)$ with a mean-squared value of

$$\langle v^2 \rangle = 4RKT\Delta f \quad \text{(Nyquist noise formula)}$$

over any bandwidth Δf, let us calculate the mean-squared capacitor voltage $\langle v_c^2 \rangle$ in the circuit over all frequencies f.

Calculation: The Thevenin equivalent circuit modeling the noisy resistor in the phasor domain is shown in the margin. The model includes a source voltage phasor $\tilde{V}$. A capacitor C with impedance $Z_c = \frac{1}{j\omega C}$ connected across the terminals of the equivalent model circuit will develop a phasor voltage

$$\tilde{V}_c = \tilde{V}\frac{Z_c}{R + Z_c} = \tilde{V}\frac{1}{1 + j\omega RC}$$

as dictated by voltage division. The mean-squared value of a co-sinusoidal oscillation $v_c(t) = \text{Re}\{V_c e^{j\omega t}\}$ with the phasor V_c would then be

$$\frac{1}{2}|\tilde{V}_c|^2 = \frac{1}{2}|\tilde{V}|^2 \frac{1}{|1 + j\omega RC|^2} \equiv \frac{1}{2}|\tilde{V}|^2 |H(f)|^2$$

where $\frac{1}{2}|\tilde{V}|^2$ is the mean-squared value of open circuit voltage $v(t)$ of the resistor and

$$|H(f)|^2 = \frac{1}{|1 + j2\pi f RC|^2} = \frac{1}{1 + (2\pi f RC)^2}$$

is a frequency dependent scaling factor — the magnitude square of the frequency response function of the circuit — between the two mean-square quantities.

Now, the mean-squared voltage output

$$\langle v^2 \rangle = 4RKT\Delta f \quad \text{(Nyquist noise formula)}$$

of the noisy resistor over a small but finite bandwidth Δf can be scaled likewise to obtain

$$\langle v_c^2 \rangle = \langle v^2 \rangle |H(f)|^2 = \frac{4RKT\Delta f}{1 + (2\pi f RC)^2},$$

the mean-squared voltage output across the capacitor over the same bandwidth provided that $|H(f)|^2$ is fairly constant over the band. For wider bands where the constancy condition is violated, use

$$\langle v_c^2 \rangle = \int_{f_1}^{f_2} \frac{4RKT}{1 + (2\pi f RC)^2} df,$$

whereas the broadband value over *all* frequencies (with $f_1 \to 0$ and $f_2 \to \infty$) is

$$\langle v_c^2 \rangle = \int_0^{\infty} \frac{4RKT}{1 + (2\pi f RC)^2} df = \frac{KT}{C},$$

a value independent of R.

For a 1 pF capacitor $K = 1.38 \times 10^{-23}$ J/K and $T = 300$ K gives an rms (root mean squared) voltage of

$$\langle v_c^2 \rangle^{1/2} = \frac{KT}{C} \approx 0.65\,\mathrm{mV},$$

easily detected in the lab.