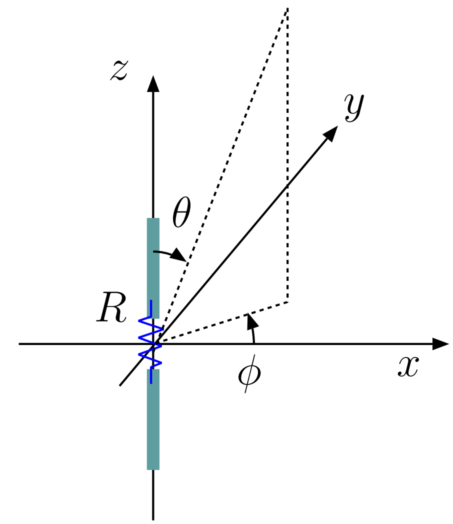
37 Effective area and reciprocity

Antenna response to multiple plane waves:



- Consider next an arbitrary antenna located at the origin with
  - a resistive termination R,
  - a gain function $G(\theta, \phi)$, and

  which is exposed to a *spectrum* of incident plane wave fields $\tilde{\mathbf{E}}_i$ at a frequency $f = \frac{\omega}{2\pi}$ arriving from directions (θ_i, ϕ_i) with

  - power densities S_i,
  - such that each $\tilde{\mathbf{E}}_i$ arriving from (θ_i, ϕ_i) is polarized identically as the field the antenna would radiate toward direction (θ_i, ϕ_i).

- Assume that each plane wave $\tilde{\mathbf{E}}_i$ will deliver an average power

$$P_r = S_i A(\theta_i, \phi_i)$$

to the resistor R one-at-a-time — where $A(\theta, \phi)$ is *by definition* the effective area of the antenna for reception — and, furthermore, the expected value of P_r extracted from the full spectrum of $\tilde{\mathbf{E}}_i$ would be

$$\langle P_r \rangle = \sum_i S_i A(\theta_i, \phi_i)$$

when phasors $\tilde{\mathbf{E}}_i$ have *random phase offsets* distributed independently between 0 and 2π radians (see the margin for an explanation).

***Incoherent power addition*: Consider a voltage**

$$v(t) = V_1 \cos(\omega t) + V_2 \cos(\omega t + \phi)$$

applied across a $1\,\Omega$ resistor where ϕ is a *random* phase shift parameter. Squaring $v(t)$ and taking its time-average it can be shown that the time average-power

$$\begin{aligned} P &= \frac{1}{T}\int_T v^2(t)dt \\ &= \frac{1}{2}V_1^2 + \frac{1}{2}V_2^2 + V_1 V_2 \cos(\phi). \end{aligned}$$

Notice the third term in P. With random ϕ we would be unable to know P, but still its *expected value* is

$$\langle P \rangle = \frac{1}{2}V_1^2 + \frac{1}{2}V_2^2,$$

which is the *incoherent sum* of the time-average power due to signals 1 and 2 *one-at-a-time*.

– This sum generalizes to an integral

$$\langle P_r \rangle = \int\int dS(\theta, \phi, f) A(\theta, \phi)$$

in case of a continuum of random incident waves over a band of frequencies f, which in turn takes a specific form

$$\langle P_r \rangle = \int\int \frac{1}{2} L(\theta, \phi, f) A(\theta, \phi) d\Omega df,$$

with

$$dS(\theta, \phi, f) = \frac{1}{2} L(\theta, \phi, f) d\Omega df = \frac{KT}{\lambda^2} d\Omega df.$$

in case of cavity radiation in thermal equilibrium — the factor $\frac{1}{2}$ is included because only one-half of the randomly polarized cavity radiance would be polarized in the same direction as the field radiated by the given antenna.

We will routinely use the approximation

$$\begin{aligned} L(f) &= \frac{2f^2}{c^2} \frac{hf}{e^{hf/KT} - 1} \\ &\approx \frac{2KT}{\lambda^2} \end{aligned}$$

in this lecture since we are concerned with radio frequencies f within narrow bands Δf over which

$$hf \ll KT.$$

Recall that radiance $L(f)$ is measured in units of W/m^2/Hz/ster.

Above relations are useful but incomplete since we have not yet identified $A(\theta, \phi)$ explicitly (although the concept was discussed last lecture) — that comes next.

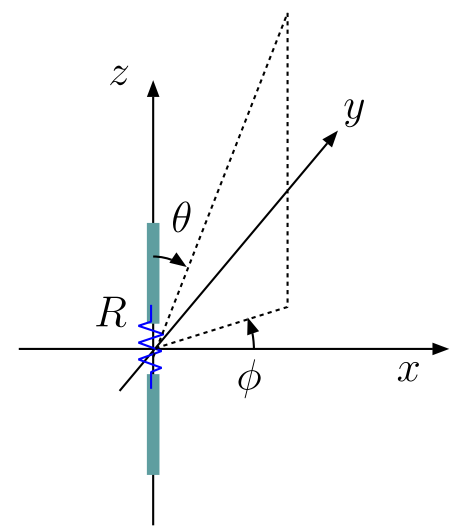
Confirmation of $A(\theta, \phi) = \frac{\lambda^2}{4\pi} G(\theta, \phi)$:

- Consider an antenna placed within a cavity, delivering *into* its resistive termination R an average power of

$$\frac{1}{2} L(\theta, \phi, f) A(\theta, \phi) = \frac{KTA(\theta, \phi)}{\lambda^2}$$

per unit frequency and solid angle of incident radiation.

  - If resistor R is in thermal equilibrium with the rest of the cavity then it would be delivering a matching amount of power back to the antenna to be radiated away (at the polarization *absorbed* — remember the $\frac{1}{2}$ factor, now we understand why).



- Since such a resistor R matched to the antenna input impedance Z_{ant} will deliver to the antenna an average power of

$$KTdf$$

over a bandwidth df, the power radiated by the antenna into a solid angle $d\Omega$ centered about a direction (θ, ϕ) will be (over the same band)

$$\frac{KTdf}{4\pi} G(\theta, \phi) \, d\Omega = \frac{KTG(\theta, \phi)}{4\pi} \, df \, d\Omega.$$

- Since thermodynamic equilibrium requires the balance of the average power transferred by the antenna between its matched termination R and the cavity radiation, we must have

$$\frac{KTA(\theta,\phi)}{\lambda^2}\,df\,d\Omega = \frac{KTG(\theta,\phi)}{4\pi}\,df\,d\Omega$$

from which it follows that

$$A(\theta,\phi) = \frac{\lambda^2}{4\pi}G(\theta,\phi).$$

This is a universal law that applies for all types of antennas (including short-dipoles, with $K_a = \frac{\lambda^2}{4\pi}$, as claimed earlier).

- The above derivation has also shown — per conditions invoked — that the *antenna effective area* $A(\theta,\phi)$ can be used to covert
  - the power density S_{inc} of only the *co-polarized* component of the incident field into
  - the power delivered P_r into a resistive termination *matching*[1] the antenna input impedance Z_{in}.

The relation

$$A(\theta,\phi) = \frac{\lambda^2}{4\pi}G(\theta,\phi)$$

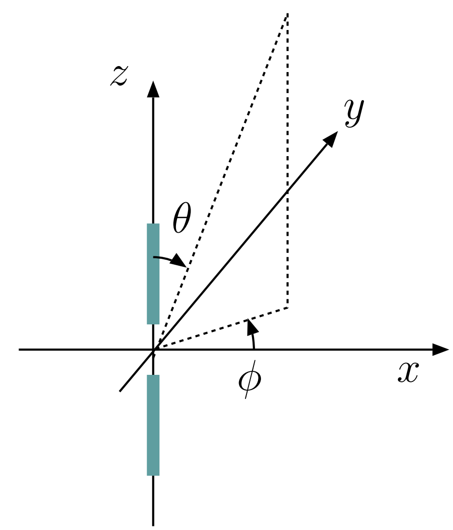
applies to all antennas universally.

The component of the incident field that is not *co*-polarized (it is then called *cross*-polarized) is not detected at all by the receiving antenna.

[1]If you are wondering about how to calculate P_r into an *unmatched* termination, the trick is to use the Thevenin equivalent of the receiving antenna implied by the matched termination in conjunction with the unmatched load. The Thevenin equivalent is obtained later in this lecture.

Based on what we have seen above (and earlier in this class), we can state that an antenna is a **transducer** for:

1. radiating its average power input P_t at its terminals to produce a far-field power density
$$S_t = \frac{P_t}{4\pi r^2} G(\theta,\phi) \text{ such that } \int G(\theta,\phi) d\Omega = 4\pi,$$
2. producing from the power density S_i of a co-polarized plane wave incident from a direction (θ,ϕ) an average *available power*
$$P_r = S_i A(\theta,\phi) \text{ such that } \int A(\theta,\phi) d\Omega = \lambda^2.$$



- Recall that **available power** of any device is the *power delivered to an impedance matched load.*

- Take advantage of the relation
$$A(\theta,\phi) = \frac{\lambda^2}{4\pi} G(\theta,\phi)$$
in transmission and reception calculations as appropriate:
  - In many cases $G(\theta,\phi)$ is already known, so the above formula is used to deduce $A(\theta,\phi)$.
  - With large 2D antennas $A_{max} = A_{physical}$, in which case
$$D = G_{max} = \frac{4\pi}{\lambda^2} A_{max} = \frac{4\pi}{\lambda^2} A_{physical} \;\Rightarrow\; \Omega = \frac{4\pi}{D} = \frac{\lambda^2}{A_{physical}}.$$