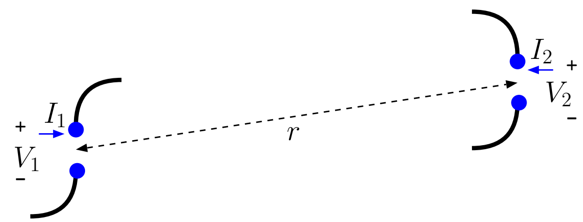
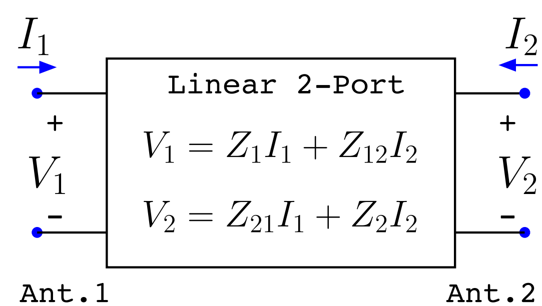
38 Equivalent circuit models of coupled antennas

- Since Maxwell's equations and the associated boundary conditions which govern the interaction of a pair of antennas with input currents $I_{1,2}$ and response voltages $V_{1,2}$ are **linear**, the *equivalent circuit model* describing the relationships of $I_{1,2}$ and $V_{1,2}$ is also **linear** and obeys a set of equations



$$\begin{aligned} V_1 &= Z_1 I_1 + Z_{12} I_2 \\ V_2 &= Z_{21} I_1 + Z_2 I_2. \end{aligned}$$

  - With I_2 set to zero, we recognize Z_1 above as the input impedance of ant.-1, while Z_{21} is a coupling impedance connecting the open circuit response voltage of ant.2 to the driver current of ant.1.
  - Likewise, Z_2 is the input impedance of ant.2 and Z_{12} is a coupling impedance from ant.2 to ant.1.



- The reciprocal behaviour of a pair of antennas separated in free space restricts the coupling impedances in the model circuit to satisfy

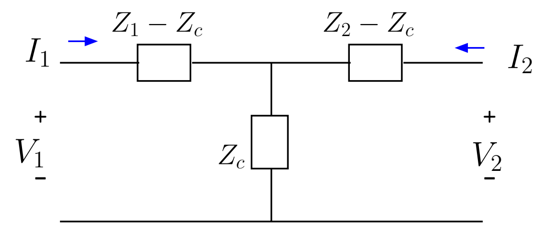
$$Z_{21} \equiv \frac{V_2}{I_1}|_{I_2=0} = Z_{12} \equiv \frac{V_1}{I_2}|_{I_1=0} \equiv Z_c$$

so that transmission gain P_t/P_r is independent of the transmission and reception roles assigned to the antennas (as verified below).

With that restriction the equivalent circuit model of the coupling simplifies as (see margin)

$$V_1 = Z_1 I_1 + Z_c I_2$$
$$V_2 = Z_c I_1 + Z_2 I_2.$$

Reciprocal circuit model for antenna coupling:



- With reference to the circuit diagram in the margin, note that the open circuit voltage at ant.2 terminals is

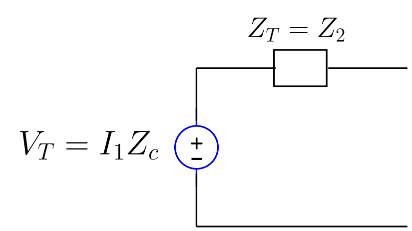
$$V_2 = I_1 Z_c \equiv V_T,$$

while the Thevenin impedance seen from ant.2 terminals is (using the source suppression method where $I_1 = 0$ case is considered)

$$Z_T = Z_2.$$

Therefore, the Thevenin equivalent circuit of ant.2 during reception takes the form shown in the margin.

Thevenin equivalent circuit of ant.2 in reception:



- From the Thevenin equivalent we identify the average power transferred by I_1 to a possible matched termination of ant.2 as

$$P_r = \frac{|V_T|^2}{8\mathrm{Re}\{Z_T\}} = \frac{|I_1|^2 |Z_c|^2}{8\mathrm{Re}\{Z_2\}}.$$

- To calculate the transmitted power P_t of ant.1 when ant.2 is terminated by matched $Z_L = Z_2^*$ (see margin), we first note that

$$V_1 \approx I_1 Z_1$$

if $|Z_c| \ll |Z_1|$, in which case ant.1 has a power input

$$P_t = \frac{1}{2}\text{Re}\{V_1 I_1^*\} \approx \frac{1}{2}|I_1|^2\text{Re}\{Z_1\}.$$

The above expressions can be combined as

$$P_r \approx \frac{1}{2}|I_1|^2\text{Re}\{Z_1\}\frac{|Z_c|^2}{4\text{Re}\{Z_1\}\text{Re}\{Z_2\}} = P_t\frac{|Z_c|^2}{4\text{Re}\{Z_1\}\text{Re}\{Z_2\}},$$

which is a reciprocal relation symmetric in Z_1 and Z_2, mimicking the symmetry of Friis transmission formula with respect to G_1 and G_2, such that

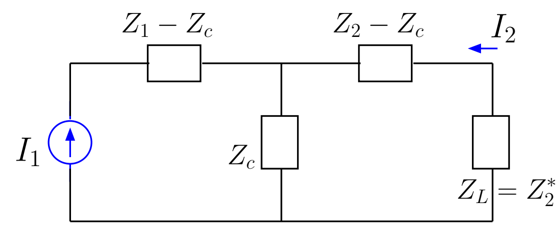
$$\frac{|Z_c|^2}{4\text{Re}\{Z_1\}\text{Re}\{Z_2\}} = \frac{\lambda^2 G_1 G_2}{(4\pi r)^2} \quad \Rightarrow \quad |Z_c|^2 = \frac{G_1\text{Re}\{Z_1\}G_2\text{Re}\{Z_2\}}{(2\pi r/\lambda)^2}$$

is implied. As shown in HW, the exact calculation of P_r, without making use of $|Z_c| \ll |Z_1|$, yields

$$P_r = P_t\frac{|Z_c|^2}{4\text{Re}\{Z_1\}\text{Re}\{Z_2\} - 2\text{Re}\{Z_c^2\}}.$$

- The above reciprocal circuit model describes the behavior of coupled antennas (of any type) separated in free space. It also applies with antennas embedded in material media (even with inhomogeneous μ, ϵ, and σ) so long as the medium is isotropic and linear as shown directly from a reciprocity condition derived from Maxwell's equations (see ECE 454 and 520). Applications of non-reciprocal antenna coupling in anisotropic media are treated in ECE 458.

Eqv. ckt. showing ant.1 in transmission and ant.2 in reception with matched load:



Note that the coupling impedance Z_c gets weaker with increasing r/λ and gets getting stronger with the product of antenna gains G_1 and G_2.

This model for Z_c and the Friis transmission formula are only valid when the antennas are placed within the far-fields (Fraunhoffer region) of each other.

Example 1: Consider a $\hat{z}$-polarized half-wave dipole antenna with an input impedance of $Z_2 = 73\,\Omega$ located at the origin. Determine the time-average power received by the antenna if it has a matched termination and the incident field has an electric field phasor

$$\tilde{\mathbf{E}}_{inc} = 120\pi\hat{z}e^{j2\pi x}\,\mathrm{V/m}.$$

Solution: The incident field has a wave vector

$$\mathbf{k} = -2\pi\hat{x}\,\mathrm{rad/m},$$

indicating the antenna sees the incident wave coming from direction $\theta = 90^\circ$ with a polarization direction $\hat{\theta}$. Since

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{|\mathbf{k}|} = \frac{2\pi}{2\pi} = 1\,\mathrm{m}$$

and

$$A(\theta,\phi) = \frac{\lambda^2}{4\pi}G(90^\circ,\phi) = \frac{1}{4\pi}\underbrace{(1.64)}_{\textbf{half-wave-dipole gain at } 90^\circ} = \frac{1.64}{4\pi}\,\mathbf{m}^2,$$

and, furthermore,

$$S_{inc} = \frac{|\tilde{\mathbf{E}}_{inc}|^2}{2\eta} = \frac{(120\pi)^2}{2\times 120\pi} = 60\pi\,\mathrm{W/m}^2,$$

it follows that

$$P_r = S_{inc}A(\theta,\phi) = 60\pi \times \frac{1.64}{4\pi} = 24.6\,\mathrm{W}$$

is the time-averaged power received by the antenna with the matched termination.

Example 2: Repeat Example 1 if the antenna is terminated by $Z_L = 36.5\,\Omega$. Also find the antenna open circuit voltage V_T.

Solution: In this case power received will be less than the available power

$$P_r = S_{inc}A(\theta,\phi) = 60\pi \times \frac{1.64}{4\pi} = 24.6\,\text{W}$$

of the antenna obtained in Example 1.

Since the the available power of an antenna with an input impedance of Z_2 is necessarily

$$P_r = \frac{|V_T|^2}{8\text{Re}\{Z_2\}}$$

in terms of the antenna open-circuit voltage phasor V_T, we obtain, using $P_r = 24.6$ W and $Z_2 = 73\,\Omega$,

$$|V_T| = \sqrt{8\text{Re}\{Z_2\}P_r} = 119.86\,\text{V}.$$

Terminating the Thevenin equivalent of the receiving half-dipole consisting of V_T and $Z_2 = 73\,\Omega$ in series with $Z_L = 36.5\,\Omega$, we deduce a load current of

$$I_L = \frac{V_T}{Z_2 + Z_L}.$$

Therefore, the load voltage is

$$V_L = I_L Z_L = \frac{V_T Z_L}{Z_2 + Z_L},$$

and the average load power is

$$\begin{aligned} P_L &= \frac{1}{2}\text{Re}\{V_L I_L^*\} = \frac{|V_T|^2\text{Re}\{Z_L\}}{2|Z_2 + Z_L|^2} = \frac{|V_T|^2}{8\text{Re}\{Z_2\}}\frac{4\text{Re}\{Z_2\}\text{Re}\{Z_L\}}{|Z_2 + Z_L|^2} \\ &= P_r\frac{4\text{Re}\{Z_2\}\text{Re}\{Z_L\}}{|Z_2 + Z_L|^2}. \end{aligned}$$

Substituting for P_r,Z_2, and Z_L, we find

$$P_L = P_r \frac{4\mathrm{Re}\{Z_2\}\mathrm{Re}\{Z_L\}}{|Z_2 + Z_L|^2} = 24.6 \frac{4 \times 73 \times 36.5}{|73 + 36.5|^2} = 24.6 \frac{4 \times 2}{9} = 21.87\,\mathrm{W}.$$